Prediction of Discretization Error using Error Transport Equation  
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Objective

- Develop a robust discretization error field prediction algorithm for a selected transport variable utilizing an error transport equation preferably on a single mesh or at most two geometrically similar meshes.

Introduction

Why Quantify discretization Errors?
- Industrial design and analysis requires quantification of all uncertainties in simulations as well as experimental uncertainty
- Discretization and modeling errors are dominant contributors to the overall simulation uncertainty
- Modeling error can not be assessed unless discretization error is quantified first.
- Quantification of discretization errors may guide grid refinement strategies to reduce it further an hence achieving greater accuracy

Theory

Let \( \phi(x,t) \) be a dependent variable governed by an equation with differential operator \( L \), and the discrete operator \( L_h \):

\[
L(\phi) = 0  \tag{1}
\]

\[
L_h(\phi_h) = 0  \tag{2}
\]

\( \phi_h \) denotes the numerical solution on a mesh represented by mesh size ‘\( h \)’. Let \( \phi_{h0} \) be the continuous function obtained by mapping of \( \phi_h \), then

\[
L_h(\phi_{h0}) = \tau_h  \tag{3}
\]

Subtracting (3) from (1) yields the error transport equation (ETE). Note: non-linear terms will yield additional error source terms. If curve fitting is done in space, time discretization errors need to be accounted separately. Taylor series expansion of the time derivative yield the error source term due to time discretization as

\[
\tau_t = \frac{dt}{v} + \frac{dt^3}{6(v^2)} + \text{Order}(dt^3) \Delta \text{vol}  \tag{4}
\]

The resulting ETE is

\[
L(\Delta \phi) = \tau_t + \tau_h \tag{5}
\]

Eq. 5 can be discretized and solved using the schemes available in the original code. For more details see Celik et al. (2016).

Verification: Sol. to non-linear Burger’s equation

Figure 1. (a) Analytical and numerical solutions and (b) errors, at t=4.0 sec, where cm represents coarse mesh, cm represent coarse mesh, U=1.0, \( \gamma = 0 \).  
Figure 2. Error at t=4.0 sec, \( \gamma = 0 \), \( \delta t = 0.1 \) and \( N_x = 61 \).

MFIX Application

Governing equations for MFIX can be found in Benyahia et al. (2007).

ETE for gas-phase volume fraction (Celik et al., 2016):

\[
\frac{\partial}{\partial t}\left( \phi \Delta u \right) = \nabla \cdot \left( \phi \lambda \mu \nabla u \right)  \tag{6}
\]

Gas-velocity error is estimated (Celik et al., 2012, 2016) from:

\[
\frac{\phi_f}{\phi_0} = \frac{\nabla \cdot \left( \phi \Delta u \right)}{\nabla \cdot \left( \phi \lambda \mu \nabla u \right)}  \tag{7a}
\]

The case considered is 1D transient ‘core annular flow’ problem presented by Benyahia et al. (2007)

Results

Figure 4. Variation of solid mass flow rate with different mesh size vs. time.  
Figure 5. True-Error vs. time, calculated by the difference between cell 160 and cell 60 cases (\( \epsilon_{160-60} \)) and error transport equation (ETE).

Figure 7. Grid convergence: difference between two successive meshes at specific location, t=80, 40, and 20 time units.

Figure 8. Error in gas-phase volume fraction.

Figure 9. Error transport equation (ETE).

Figure 10. Time averaged and temporal errors in gas phase volume fraction, calculated by the difference between cell 160 and cell 60 cases (\( \epsilon_{160-60} \)) and error transport equation (ETE).

Figure 11. True-Error vs. time, calculated by the difference between cell 160 and cell 60 cases (\( \epsilon_{160-60} \)) and error transport equation (ETE).

Figure 12. Time averaged and temporal errors in gas phase volume fraction, calculated by the difference between cell 160 and cell 60 cases (\( \epsilon_{160-60} \)) and error transport equation (ETE).

Blue curves show the error transport equation can predict the time averaged errors accurately. Furthermore, the transient of the error can also be predicted by the ETE with reasonable accuracy.

Conclusions

- An error transport equation (ETE) is formulated for applications in conjunction with multi-phase flow codes such as MFIX to predict the discretization error field distribution.
- Verification against exact solution of Burgers equation with ‘smooth’ transient solutions yielded very good results on single-grid simulations.
- Application to 1D transient/core annular flow case using MFIX showed encouraging results when two simulations on two different meshes are utilized in evaluation of the error source terms in the ETE for the gas-phase volume fraction.
- Future work will focus on extending the same method to other variables and more realistic 2D transient flow cases.

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References