NOMENCLATURE

$\mathbf{a}$	acceleration vector
$\mathbf{a}_{vm}$	relative acceleration between components
a	radius of the sphere
A_i	interaction coefficients, i = 1 to 5
$\mathbf{b}$	body force vector
$\mathbf{D}$	symmetric part of the velocity gradient
$\mathbf{f}_I$	interaction force vector
F	volume fraction dependence of drag
$\mathbf{I}$	identity tensor
$\mathbf{L}$	gradient of velocity vector
P	fluid pressure
R	radius of the pipe
Re	Reynolds number
$\mathbf{T}$	stress tensor
U_0	reference velocity
$\mathbf{v}$	velocity vector
$\mathbf{W}$	spin tensor
$\mathbf{x}$	position vector

Greek Letters

λ_f	second coefficient of fluid viscosity
μ	first coefficient of fluid viscosity
ν	volume fraction of the solid
ρ	density
ρ_0	reference density
ϕ	volume fraction of fluid

Subscripts

1,f	referring to the fluid phase
2, s	referring to the solid phase
m	referring to the mixture

Superscripts

T	transpose
*	dimensionless quantity

Other Symbols

div	divergence operator
∇	gradient operator
tr	trace of a tensor
$\otimes$	outer product
	dot product

(Lefebvre [46], Sirignano [72], Crowe et al. [17], Clift et al. [16], Sadhal et al. [66]). This is known as the 'Dilute phase or Lagrangean approach'. In the second approach, the two phases are interacting with each other to such an extent that each phase (or each component) directly influences the motion and the behaviour of the other phase. This is known as the "Dense phase approach," sometimes also called the Eulerian (or the two-fluid) approach. This method is used extensively in fluidization (cf. Davidson et al. [18], Gidaspow [27]), gas-solid flows (cf. Fan and Zhu [26]), pneumatic conveying (cf. Marcus et al [48]), suspensions (cf. Govier and Aziz [28]), Ungarish [81]), and is described for a variety of applications in general textbooks such as Soo [73], Meyer [57], Roco [64], Wallis [82], Hestroni [36], Joseph and Schaeffer [44], Afgan [3], Larson [45]...

The large number of articles published concerning multiphase flows typically employ one of the two continuum theories developed to describe such situations: Mixture theory (or the theory of interacting continua) or Averaging (cf. Ishii [39]). Both approaches are based on the underlying assumption that each

phase may be mathematically described as a continuum. Mixture Theory was first formulated by Truesdell [76]. It provides a means for studying the motions of bodies made up of several constituents by generalizing the equations and principles of the mechanics of a single continuum. The fundamental assumption in this theory is that at any instant of time, every point in space is occupied by one particle from each constituent. Since the approach that we have taken in this study and our other studies (cf. Johnson et al [43], Massoudi et al [51], Massoudi and Lakshmana Rao [53]) is based on Mixture Theory, a brief review of the notation and basic equations of Mixture Theory is presented in the next section. More detailed information, including an account of the historical development, is available in the articles by Adkins [1, 2], Green and Naghdi [29, 30, 31], Atkin and Craine [6, 7], Bowen [12], and Bedford and Drumheller [10], and in the books by Truesdell [77], Dobran [21], and Rajagopal and Tao [62]. We consider the mixture of fluid and solid particles to be a purely mechanical system in which thermal, electromagnetic, and chemical reactions are ignored [these effects can of course be included in more general theories of mixtures].

The boundary value problem that we are interested in, and which is extremely important from an experimental point of view, is the flow of a mixture between two rotating cylinders. Massoudi and Johnson [52] investigated this (Couette) flow in a recent paper. In that paper, they looked at some of the effects of the material parameters, such as the viscosity of the fluid, and they focused to some extent, on the importance of the 'slip-shear' lift. In the present investigation, we will give the results for other material parameters, such as the viscosity of the solid phase, and the 'spin' lift contribution. This latter term introduces and adds a strong non-linearity in the equations. We will also derive the expression for the traction vector (related to the wall frictional effects), and the torque on the walls of the cylinder (a quantity of interest to experimentalists).

GOVERNING EQUATIONS

In order to show where and how constitutive relations are needed in multiphase flow problems, we present the basic equations of motion. In the absence of any chemical, thermal, or electro-magnetic effects, these equations are the conservation laws for mass, linear momentum, and angular momentum.

Conservation of Mass

Assuming no interconversion of mass between the two constituents, conservation of mass for the fluid and solid are, respectively:

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{v}_1) = 0, \tag{1}$$

and:

$$\frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathbf{v}_2) = 0. \tag{2}$$

Also, ρ_1 and ρ_2 are the bulk densities of the mixture components given by:

$$\rho_1 = \phi \rho_f, \quad \rho_2 = \nu \rho_s, \tag{3}$$

where ρ_f is the density of the pure fluid, ρ_s is the density of the solid grains, ν is the volume fraction of the solid component, and ϕ is the volume fraction of the fluid. For a saturated mixture $\phi = 1 - \nu$. The mixture density, ρ_m is given by:

$$\rho_m = \rho_1 + \rho_2, \tag{4}$$

and the mean velocity $\mathbf{v}$ of the mixture is defined by:

$$\rho_m \mathbf{v} = \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2. \tag{5}$$

Conservation of Linear Momentum

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the partial stress tensors of the fluid S_1 and the solid S_2, respectively. Then the balance of linear momentum for the fluid and solid are given by:

$$\rho_1 \frac{D_1 \mathbf{v}_1}{Dt} = \operatorname{div} \mathbf{T}_1 + \rho_1 \mathbf{b}_1 + \mathbf{f}_I, \tag{6}$$

$$\rho_2 \frac{D_2 \mathbf{v}_2}{Dt} = \operatorname{div} \mathbf{T}_2 + \rho_2 \mathbf{b}_2 - \mathbf{f}_I, \tag{7}$$

where $\mathbf{b}$ represents the body force, and $\mathbf{f}_I$ represents the mechanical interaction (local exchange of momentum) between the components.

Conservation of Angular Momentum

The balance of moment of momentum implies that:

$$\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{T}_1^T + \mathbf{T}_2^T \tag{8}$$

The partial stresses however need not be symmetric.

CONSTITUTIVE RELATIONS

Eqns (1,2) and (6,7) represent the basic governing equations for two-phase flows consisting of a large number of solid particles or gas bubbles suspended in a fluid medium (such as air or water). Looking at these equations, it is clear that we need constitutive relations for $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{f}_1$. The structure of these equations is general, and they can be applied to any two-phase flow systems, where the dispersed phase is dense enough that it can be treated as a continuum. Since we are interested in the flow of solid particles entrained in a viscous fluid, we use the mechanics of granular materials to use an expression for the stress tensor of the solid phase. In general, the constitutive expressions for $\mathbf{T}_f$ and $\mathbf{T}_s$ depend on the kinematical quantities associated with both the constituents. However, we assume that $\mathbf{T}_s$ and $\mathbf{T}_f$ depend only on the kinematical quantities associated with the solid and fluid, respectively. This assumption is sometimes called "the principle of phase separation" and was first put forward by Adkins [1, 2].

In the majority of fluid-solid mixtures, the fluid is either a gas or water. Therefore, it is appropriate to assume that the fluid behaves as a linearly viscous fluid, whose constitutive equation is

$$\mathbf{T}_f = \left[-p(\rho_1) + \lambda_f(\rho_1)\,\mathrm{tr}\,\mathbf{D}_1\right]\mathbf{I} + 2\mu_f(\rho_1)\mathbf{D}_1 \tag{9}$$

where p is the fluid pressure, λ_f and μ_f are the viscosities, $\mathbf{D}_1$ is the symmetric part of the velocity gradient of the fluid, and $\mathbf{I}$ is the identity tensor. If the fluid is incompressible, then p is one of the unknown quantities in the problem that would have to be calculated. If the fluid is compressible, an equation of state is needed. In general, p, λ_f, and μ_f are functions of ρ_1.

There are basically two different ways of deriving a constitutive relation for the stress tensor of granular materials - the continuum approach and the statistical approach. For a general treatment of the flow of granular materials, we refer the reader to the review article by Hutter and Rajagopal [38], and the book by Mehta [56]. We use the continuum approach in our analysis. We assume that the stress tensor for a granular material is given by Rajagopal and Massoudi [61],

$$\begin{aligned}\mathbf{T}_s = [&\hat{\beta}_0(\rho_2) + \hat{\beta}_1(\rho_2)\operatorname{grad}\rho_2 \bullet \operatorname{grad}\rho_2 \\ &+ \hat{\beta}_2(\rho_2)\operatorname{tr}\mathbf{D}_2]\ \mathbf{I} + \hat{\beta}_3(\rho_2)\mathbf{D}_2 \\ &+ \hat{\beta}_4(\rho_2)\operatorname{grad}\rho_2 \otimes \operatorname{grad}\rho_2,\end{aligned} \tag{10}$$

where $\bullet$ denotes the scalar product of two vectors and $\otimes$ denotes the outer product of two vectors. The spherical part of the stress in Eq. (10) can be interpreted as the solid pressure p_s. The material moduli $\hat{\beta}_1$ and $\hat{\beta}_4$ are material parameters that reflect the distribution of the granular particles, and $\hat{\beta}_0$ plays a role analogous to pressure in a compressible fluid and is given by an equation of state. The material modulus $\hat{\beta}_2$ is a viscosity akin to the second coefficient of viscosity in a compressible fluid and $\hat{\beta}_3$ denotes the viscosity (i.e., the resistance of the material to flow) of the granular solids.

Following, Rajagopal and Massoudi [61] we assume that the material parameters have the structure

$$\begin{aligned}\beta_0(\nu) &= k\nu, \\ \beta_1(\nu) &= \beta_{10} + \beta_{11}\nu + \beta_{12}\nu^2, \\ \beta_2(\nu) &= \beta_{20} + \beta_{21}\nu + \beta_{22}\nu^2, \\ \beta_3(\nu) &= \beta_{30} + \beta_{31}\nu + \beta_{32}\nu^2, \\ \beta_4(\nu) &= \beta_{40} + \beta_{41}\nu + \beta_{42}\nu^2.\end{aligned} \tag{11}$$

The above representation can be viewed as a Taylor series approximation for the material parameters. Further restrictions on the coefficients can be obtained by using the following argument. Since the stress should vanish as $\nu \to 0$, we conclude that

$$\beta_{30} = \beta_{20} = 0 \tag{12}$$

Which is really a principle of the 'limiting case'. The rationale for the structure given above can also be found in Rajagopal and Massoudi [61] who also show that

$$k < 0 \tag{13}$$

as compression should lead to densification of the material.

A mixture stress tensor is defined as (cf. Green and Naghdi [31])

$$\mathbf{T}_m = \mathbf{T}_1 + \mathbf{T}_2 \tag{14}$$

Where

$$\mathbf{T}_1 = (1-\nu)\mathbf{T}_f \text{ and } \mathbf{T}_2 = \mathbf{T}_s \tag{15}$$

so that the mixture stress tensor reduces to that of a pure fluid as $\nu \to 0$ and to that of a granular material as $\phi \to 0$. $\mathbf{T}_2$ may also be written as $\mathbf{T}_2 = \nu\hat{\mathbf{T}}_s$, where $\hat{\mathbf{T}}_s$ may be thought of as representing the stress tensor for some (quite densely packed) reference configuration of the granular material.

The mechanical interaction between the mixture components, $\mathbf{f}_I$, is written as (cf. Johnson et al. [42]),

$$\begin{aligned} \mathbf{f}_I = {} & A_1 \operatorname{grad} \nu + A_2 F(\nu)(\mathbf{v}_2 - \mathbf{v}_1) \\ & + A_3 \nu (2 \operatorname{tr} \mathbf{D}_1^2)^{-1/4} \mathbf{D}_1 (\mathbf{v}_2 - \mathbf{v}_1) \\ & + A_4 \nu (\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{v}_2 - \mathbf{v}_1) + A_5 \mathbf{a}_{vm} \end{aligned} \tag{16}$$

where $\mathbf{W}_1$ and $\mathbf{W}_2$ are the spin tensors for the fluid and solid phases, respectively, $\mathbf{a}_{vm}$ is a properly frame invariant measure of the relative acceleration between the mixture components and F (ν) represents the dependence of the drag coefficient on the volume fraction. The terms in Eq. (16) reflect the presence of density gradients,[1] drag, "slip-shear" lift, "spin" lift, and virtual mass, respectively (The effect of pressure gradient within the context of mixture theory was incorporated in this Eqn by Briggs et al [15]). Müller [58] indicates that a term of the form A_1 *grad* ν must be included in the interactions in order to get well-posed problems. The term multiplying A_3 is a generalization of Saffman's [67, 68] single particle result first proposed in this form by McTigue et al. [55]. The term multiplying A_4 is a generalization of the result by Rubinow and Keller [65] [spin-lift]. As we explained earlier, within the framework of mixture theory developed here, we cannot directly account for particle size, or different particle sizes, or particle shape, or surface roughness, or . . . However, since the coefficients in Eqn (16)

[1] The actual form of this interaction should include the terms α_1 grad $\rho_1 + \alpha_2$ grad ρ_2 where α_1 and α_2 are constants. If we assume that the system is a saturated mixture with incompressible components, this expression simplifies to A_1 grad ν where $A_1 = \alpha_2 - \alpha_1$. Since no information concerning the coefficients α_1 and α_2 is available and a term of the same form arises from the granular solid stress tensor, this term will be neglected in the present work.

are determined either by performing simple experiments on an assembly of known size and known shape particles, or by extrapolating and extending the results of a single particle, material or geometrical properties can enter in the constitutive relation through these coefficients. Therefore, we have, for example, (cf. Johnson et al. [42] for details):

$$
\begin{aligned}
A_2 &= \frac{9}{2}\frac{\mu_f}{a^2} \\
A_3 &= \frac{3(6.46)}{4\pi}\frac{\rho_f^{1/2}\mu_f^{1/2}}{a} \\
A_4 &= \frac{3}{4}\rho_f \\
A_5 &= 2\frac{\pi}{3}a^3\nu\frac{1+2\nu}{1-\nu} \\
F(\nu) &= \nu(1+6.55\nu)
\end{aligned}
\tag{17}
$$

where a is the particle radius. Most of these coefficients are given either as a generalization of single particle result or as a result of some other limiting conditions. For example, the expression for F (ν) in this form, which is given by Drew [22], is based on the result of Batchelor's [9] study of sedimentation of spheres (for details see Johnson et al. [42]). In a recent study Massoudi and Lakshmana Rao [53] have looked at a different correlation in their study of a mixture in a vertical channel. We have neglected the Basset force for the present because its significance in flows with many particles has not been established. Also, if the flow is steady, virtual mass effects may be neglected. If the particles are non-spherical, such as fibers, then the directionality may become an important element, and in that case we need to use (or develop) the constitutive relations which take the microstructure and directionality, i.e., the anisotropic nature of the material, into account (cf. Ericksen [24, 25], Leslie [47]). The way in which we use Eqn (16) along with Eqn (17) in our studies is that since the governing equations are made dimensionless, the material parameters such as viscosity and density along with the numerical coefficients appearing in these correlations are absorbed in the dimensionless numbers. Therefore, the actual numerical values of these experimentally obtained coefficients do not enter into our calculations directly; instead we perform a parametric study for a range of these dimensionless numbers.

In majority of multiphase studies, lift forces are neglected. Massoudi [53] discusses the importance of these forces and through a simple order of magnitude

analysis concludes that these forces, especially in the regions of high velocity gradients (such as swirling flows), cannot be assumed to be negligible *a priori*. In the next section, we give the background and the rationale for the specific forms of the lift forces shown in Eqn (16). Massoudi [54] provides a brief review of the interaction mechanisms and indicates that Eqn (16) is a reasonably general and appropriately frame-invariant model, which can be used as a constitutive relation for many applications.

LIFT FORCES

It is observed (cf. Segre and Silberberg [69, 70, 71]) that spheres in laminar Poiseuille flow through a pipe (at low Re) accumulate in an annulus some distance from the tube axis. Following the initial observations, a number of investigators verify this 'tubular pinch' effect and attempt to explain the lateral (or lift) force acting on the spheres (cf. Denson et al. [20], Eichorn and Small [23], Jeffrey and Pearson [41], Oliver [59], Repetti and Leonard [63]). Though some authors attempt to explain the radial migrations in terms of particle spin (i.e., Magnus force), spheres prevented from spinning also reached equilibrium positions between wall and centerline (cf. Denson et al. [20], Oliver [59], Repetti and Leonard [63]). Brenner [13] gives a detail review of this issue.

Bretherton [14] shows that, based on creeping flow equations, there is no lateral force on a single rigid particle in a unidirectional flow. Saffman [67, 68] deduces that, since experimental results contradicted this conclusion, inertial effects must be involved. Saffman obtains a result for 'slip-shear' lift on a particle at low Reynolds number analogous to a result derived earlier for 'spin' lift by Rubinow and Keller [65]. Saffman [67] includes particle spin in his analysis and shows that under circumstances where his results and Rubinow and Keller's result strictly apply, the 'shear' lift dominates the 'spin' lift (see also Boothroyd [11]). The lift force Saffman derives is normal to the slip vector and the spin vector of the particle. If the particle lags the fluid, the lift will move the particle toward the faster adjacent fluid and vice versa if the particle leads the fluid. Aoki et al. [4] claim that for larger particle Reynolds numbers (i.e., $Re_p \geq 1$) the Magnus effect explains the observed particle migrations. Halow [34] discusses the importance of lift forces in a general context of turbulent pipe flows. Ho and Leal [37] calculate another form of lift force on a single particle in a channel. This force is apparently a result of wall effects. Auton (1987) studies the lift force due to a weak shear flow of an inviscid fluid. A detailed analysis, including experimental observation of lift forces in Couette systems is given by Halow and Wills [32, 33]. For an updated review of Saffman's contribution to this field, we refer the reader to the recent article by Stone [74].

Boothroyd [11] compares the relative magnitudes of the 'slip-shear' lift derived by Saffman [67, 68] and the 'spin' lift derived by Rubinow and Keller [65]. The 'slip-shear' lift on a single particle is written as:

$$f_{Ls} = 6.46\,\mu_f^{\frac{1}{2}}\rho_f^{\frac{1}{2}}a^2\left|\frac{dv}{dy}\right|^{\frac{1}{2}}(u - v) \tag{18}$$

where u is the particle velocity and v is the fluid velocity. Even though Saffman has retained the inertial terms of the Navier-Stokes equations in his analysis, the flow is not inertailly dominated. Decker [19] points out that Saffman's analysis of lift force includes the following implicit assumptions: the flow is uniform and parallel, the slip velocity is parallel to the plane of fluid shear, the shear or velocity gradients of the fluid are linear and the particle spin vector lies in the plane of fluid shear, but is normal to the slip vector. To obtain the expression for the 'spin' lift, we need to discuss Magnus force.

If we consider a body travelling through a fluid such that its axis of rotation does not coincide with its direction of translation (i.e., its path or trajectory), then the body will experience a lift force in a direction perpendicular to the plane defined by its axis of rotation and its trajectory. The magnitude of this force, known as the Magnus force, depends upon spin rate, velocity, and shape of the body. The so-called 'Magnus effect,' in which the force just described causes a deflection perpendicular to the flight path, was first investigated in relation to deflection of projectiles (such as tennis balls or baseballs). A historical review is given by Barkla and Auchterlonie [8]. A relatively large amount of work ahs was done concerning the Magnus force acting on spinning objects of large size (cf. Swanson [75]). The theoretical analyses for these situations, however, are based on inviscid flow and therefore inappropriate for predicting lift on small particles at low Reynolds numbers where viscous effects are expected to dominate. Generally, lift coefficients are measured empirically; though experimental work has also been primarily concerned with large objects at high Reynolds numbers. Tsuji et al. [80] perform experiments on small spheres (5 mm dia.) at Reynolds numbers as low as 550. Rubinow and Keller [65] obtain an analytical expression for the lift force on a small spinning sphere. They used an expression valid for small values of the particle Reynolds number to calculate the transverse force on the sphere. A possible generalization of their result for the 'spin' lift is written as:

$$\mathbf{f}_{Lm} = \pi\rho_f a^3\overline{\Omega} \times (\mathbf{u} - \mathbf{v}) \tag{19}$$

where a is the radius of the sphere, and $\overline{\Omega}$ is the angular velocity of the particle (a vector quantity). Note that $\mathbf{f}_{Lm}$ acts in the same direction (perpendicular to the slip velocity) as f_{Ls}. If we assume that the particle angular velocity is given by:

$$\Omega = \frac{1}{2}\frac{dv}{dy} \tag{20}$$

then Equation (19) may be written:

$$f_{Lm} = \pi \rho_f a^3 \frac{1}{2}\frac{dv}{dy}(u - v) \tag{21}$$

Forming the ratio of f_{Lm} to f_{Ls} for flow in a pipe yields:

$$\frac{f_{Lm}}{f_{Ls}} = 0.24\frac{a}{R}\left[\frac{Re_f R}{U}\frac{dv}{dy}\right]^{\frac{1}{2}} \tag{22}$$

where R is the radius of the pipe, U is a reference velocity, and Re_f is the pipe Reynolds number based on R and U. Boothroyd concludes from this expression that quite often f_{Ls} is much larger than f_{Lm}.

Magnus (or spin) force is generally neglected in models of two-phase flow on the basis of order of magnitude arguments (cf. Saffman [67], Boothroyd [11]). These arguments, however, are only strictly valid for the low Reynolds number (and otherwise idealized) situations used by Saffman [67] and Rubinow and Keller [65] to derive their analytical results. At least one experimental study (cf. Aoki et al. [4]) has stated that at higher Reynolds numbers particle migration can be explained by Magnus force. It is by no means certain, on the basis of these arguments, that Magnus force can be neglected in two-phase flows. We generalize Rubinow and Keller's results, in the same manner as Drew generalizes Saffman's, to get:

$$\mathbf{f}_{Lm} = \frac{3}{4}\nu \rho_f (\mathbf{W}_s - \mathbf{W}_f)(\mathbf{u} - \mathbf{v}) \tag{23}$$

An interaction force of this from was included in the mixture equations employed by Atkin and Craine [6]. McTigue et al. [55] clarifies Drew's [22] generalization of Saffman's expression for the lift force to obtain:

$$\mathbf{f}_{Ls} = \frac{3(6.46)}{4\pi}\frac{\rho_f^{\frac{1}{2}}\mu_f^{\frac{1}{2}}\nu}{a}(\mathrm{tr}\,\mathbf{D}_f^{\,2})^{-\frac{1}{4}}\mathbf{D}_f(\mathbf{u} - \mathbf{v}) \tag{24}$$

Ho and Leal [37] use an asymptotic expansion to calculate the lift on a neutrally buoyant spherical particle in a simple shear or a Poiseuille flow. Their analysis utilizes the concept of an 'undisturbed' flow (that is, the flow field that would exist if the particle were not present) and arrives at a lateral force on a particle given by:

$$f_L = \kappa^2 \mathrm{Re}[\beta G_1(s) + \beta\gamma G_2(s)] \tag{25}$$

This expression is valid for all undisturbed flow fields of the form $\alpha + \beta z + \gamma z^2$. Here, d is the distance between walls, a is some characteristic length scale (such as particle radius), and $\kappa = \frac{a}{d}$. α, β, and γ are constants, G_1 (s) and G_2 (s) are functions of the flow geometry numerically calculated and tabulated by the authors, and $s = \frac{z}{d}$ is the dimensionless z coordinate. The Reynolds number, Re, is defined by:

$$\mathrm{Re} = \frac{\rho_f \dot{v}_c a}{\mu_f} \tag{26}$$

where ρ_f and μ_f are the density and viscosity of the fluid respectively and $\dot{v}_c$ is the velocity scale (i.e., mean flow velocity).

McTigue et al. [55] cite the above result in using an expression for lift that includes a term of the form:

$$\mathbf{f}_L = L\, \mathbf{D}_f \operatorname{div} \mathbf{D}_f \tag{27}$$

where L is a constant coefficient and $\mathbf{D}_f$ is defined by:

$$\mathbf{D}_f = \frac{1}{2}\left[\operatorname{grad} \mathbf{v}_f + (\operatorname{grad} \mathbf{v}_f)^T\right] \tag{28}$$

They state that the term given by Equation (28) is a generalization of the lift force (Equation 25) derived by Ho and Leal [37].

We hope that this brief review of lift forces would provide some justification for including, at least two of these lift forces, namely the 'shear-lift,' and the 'spin-lift' in our study. Equations (23) and (24) are basically incorporated into Eqn (16)

with coefficients A_3 and A_4. In the next section, we will look at the boundary value problem for the case of flow between two rotating cylinders.

FLOW BETWEEN TWO ROTATING CYLINDERS

We consider the Couette flow of a mixture confined to the gap between two vertical, infinitely long, concentric rotating cylinders of circular cross section. Let the wall of the interior cylinder be located at r=1 and the outer cylinder at r_0 (r_0>1). If the flow is steady and laminar, the velocity profiles and solids distribution can be assumed to have the form

$$\mathbf{V}_1 = V(r)\mathbf{e}_\theta,\ \mathbf{V}_2 = U(r)\mathbf{e}_\theta,\ \nu = \nu(r) \tag{29}$$

In a cylindrical coordinate system, the balance of mass equations are automatically satisfied by these assumptions. Virtual mass effects are not important for this steady problem, and the body forces are neglected.

The momentum balances given by Eqs. (6) and (7), using Eqns (9)-(13), and (16)-(17), are greatly simplified by the kinematical assumptions given by Eqn (29), and may be written in component form as

$$(1-\nu)(V'' + \frac{1}{r}V' - \frac{1}{r^2}V) - \nu'(V' - \frac{1}{r}V) + D_1\,\mathcal{F}(\nu)(U-V) = 0 \tag{30}$$

$$\left\{(\nu+\nu^2)(U'' + \frac{1}{r}U' - \frac{1}{r^2}U) + (2\nu+1)\nu'(U' - \frac{1}{r}U)\right\} - D_2\,\mathcal{F}(\nu)(U-V) = 0 \tag{31}$$

$$\begin{aligned} &P\nu\frac{1}{r}U^2 + B\nu' + \left\{2(1+\nu+\nu^2)\nu'\nu'' + (2\nu+1)(\nu')^3\right\} + E\frac{1}{r}(\nu')^2(1+\nu+\nu^2) - \\ &L_1\nu\left|V' - \frac{1}{r}V\right|^{-1/2}(V' - \frac{1}{r}V)(U-V) - L_2\nu(V' + \frac{1}{r}V - U' - \frac{1}{r}U)(U-V) = 0 \end{aligned} \tag{32}$$

where the prime denotes a derivative with respect to r.

In order to solve the above system of equations the following definitions are introduced for simplicity:

$$\begin{aligned} &D_1 = C_2 Re,\ D_2 = \frac{2C_2}{B_2},\ B = \frac{B_0}{B_1+B_4},\ E = \frac{B_4}{B_1+B_4} \\ &P = \frac{\rho_s}{(B_1+B_4)},\ L_1 = \frac{-C_3}{2(B_1+B_4)},\ L_2 = \frac{-C_4}{2(B_1+B_4)} \end{aligned} \tag{33}$$

These dimensionless parameters have the following physical interpretations. D_1 is the ratio of the drag forces exerted on the fluid by the solid to the viscous forces within the fluid phase. D_2 is the ratio of the drag force exerted on the solid particles by the fluid to the viscous forces within the assembly of the solid particles. B and E are combinations of the material coefficients from the granular solid constitutive equation; they affect the distribution of the particles in the flow but do not directly influence the velocity profiles of the either the fluid or the solid. They are also related to the normal stress coefficients responsible for the dilatancy phenomenon in granular materials (Massoudi and Mehrabadi [54]). P is the ratio of the centrifugal forces to the forces generated by particle-particle interactions within the granular materials. L_1 incorporates the effect of slip-shear lift into the equation; L_2 is the spin-lift coefficient. The fluid and solid velocity fields exert an effect on the distribution of the granular materials only when $L_1 \neq 0$ and/or $L_2 \neq 0$.The drag coefficient, Reynolds number, and material viscosities must be greater than zero; therefore, D_1 and D_2 are positive. The lift coefficient L_1 and L_2 are less than zero. P is restricted to be negative implying $B_1 + B_4 < 0$. Since B_0 is also less than zero (Rajagopal and Massoudi, [61]), B must be greater than zero.

Boundary Conditions

Adherence boundary conditions are imposed on both the constituents at both cylinder walls:

$$\begin{aligned} U(1) &= V(1) = W_i \\ U(R_0) &= V(R_0) = W_0 \end{aligned} \tag{34}$$

Where W_i is the velocity of the inner cylinder and W_o is the velocity of the outer cylinder. The volume fraction at one of the boundaries is specified,

$$\nu = \nu_{set} \tag{35}$$

The second boundary condition for the volume fraction is defined through

$$N = 2\pi \int_1^{R_0} \nu \, r \, dr \tag{36}$$

where N is specified.

RESULTS AND DISCUSSIONS

Equations (30)-(32) are solved using the collocation code COLSYS (Ascher et al. [5]). Since the equations are in their non-dimensional forms, we will present the results for velocity and volume fraction profiles for a selected range of the dimensionless numbers. From a practical point of view, one of the cylinders can be held stationary and the other rotating, or they can be rotated in the opposite directions. Also, one could glue some particles at the wall or allow slip at the wall. The torque is an important quantity which is related to the traction on the inner (or outer) cylinder. We will present these equations at the end of this section.

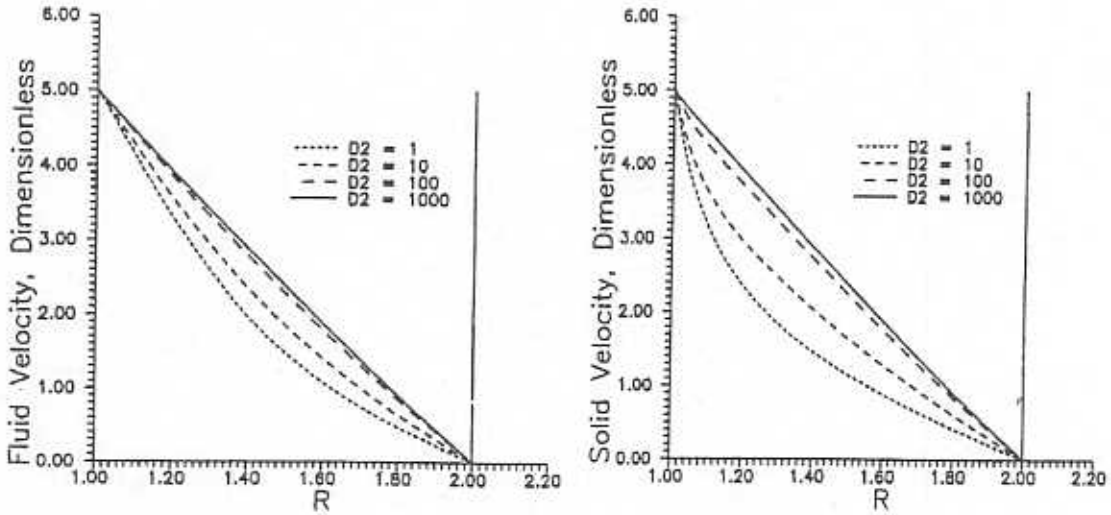

Figure 1. Couette Flow with rotating inner cylinder: Effect of D_2 on fluid(the left figure) and solid (the right figure) velocity profiles; $D_1 = 10$, $B = 4$, $E = 4$, $L_1 = L_2 = 0$, $P = -1$, $N = 0.4$.

Some of the results, a few selected cases, where presented in Massoudi and Johnson [52]. In the present paper, we will focus on the dimensionless numbers which were not considered in the previous study; effects of D_1, E, and L_1 are given in that study. Figures 1 through 5 are obtained for W_i=5 and W_o=0. Figures 6 though 10 are the corresponding results for W_i=0 and W_o=5. We can see that increasing D_2 results in an increase in fluid velocity and a corresponding increase in the solid velocity (see Fig. 1). Physically, an increase in the value of D_2 corresponds to either an increase in the drag coefficient or a decrease in the non-dimensional number B_3 (related to the viscosity of the granular materials). Figure 2 shows the fluid and solid velocities for two different values of D_2.

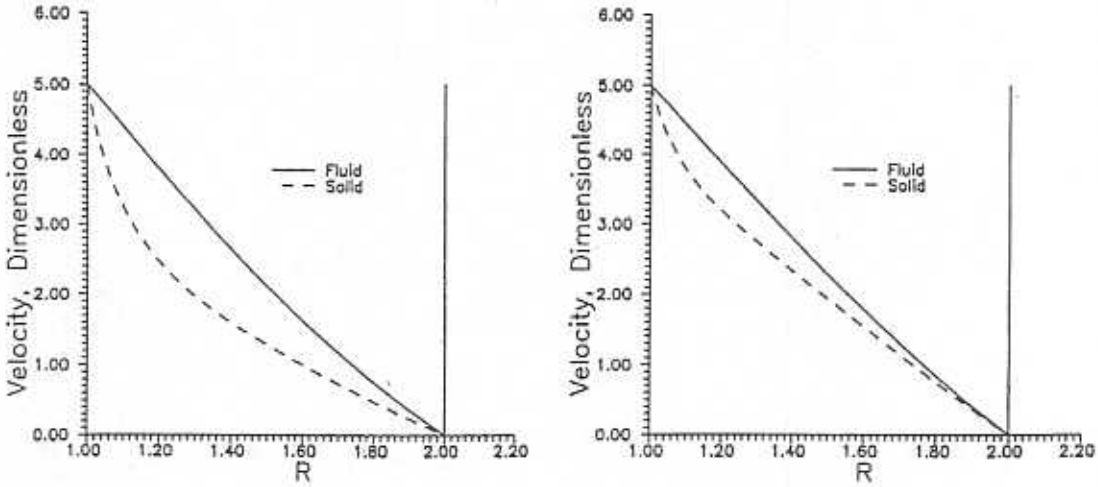

Figure 2. Couette Flow with rotating inner cylinder: Comparison of fluid and solid velocity profiles; a) $D_2 = 1$(top), and (b) $D_2 = 10$ (bottom); $D_1 = 1$, B=4, E=4, $L_1 = L_2 = 0$, P=-1, N=0.4.

Figure 3 shows plots of volume fraction versus r for some representative values of B (with $L_1 = L_2 = 0$) and E=4). Note that an integral condition is used and the areas under each of the curves are the same, although the boundary

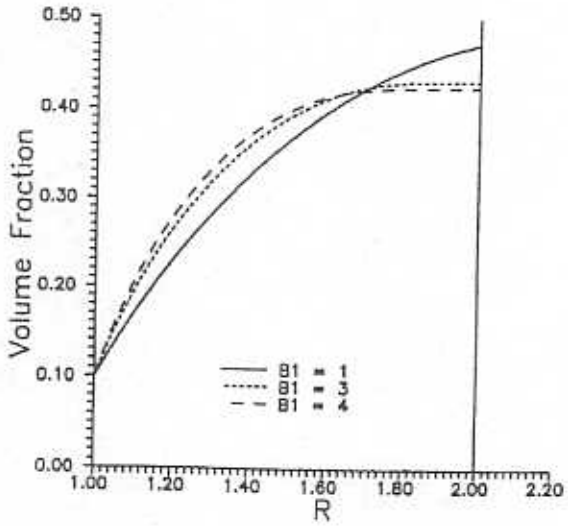

Figure 3. Couette Flow with rotating inner cylinder: Effect of B on volume fraction profiles; $D_2 = 1$; $D_1 = 10$, E = 4, $L_1 = L_2 = 0$, P = -1, N = 0.4

condition at the outer wall is different for each curve. Figure 4 shows the effect of P on the velocity profiles. A larger value of P results in a greater portion of the granular solid collecting near the outer cylinder wall (not shown here). Physically, an increase in the magnitude of P corresponds to an increase in ρ_s , the density of the granular solid. Note in Fig 4 that the different solid distribution has relatively

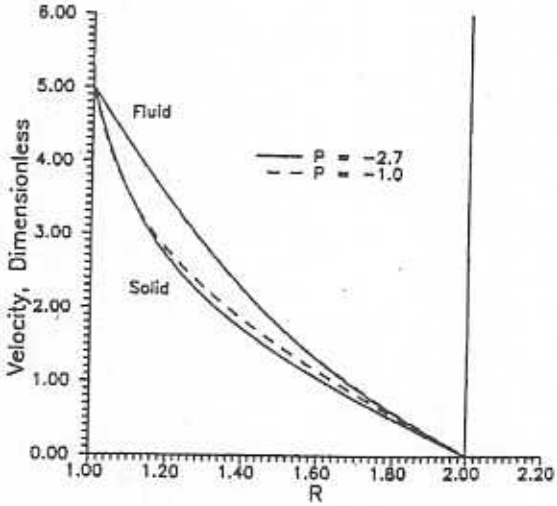

Figure 4. Couette Flow with rotating inner cylinder: Effect of P on velocity profiles; $D_2 = 5$; $D_1 = 10$, $B = 4$, $E = 4$, $L_1 = L_2 = 0$, $N = 0.3538$.

little effect on the velocity profiles. Figure 5 shows a representative result from a parametric study of the effects of 'spin' lift, L_2 , on the volume fraction. Spin lift and slip-shear lift appear to have essentially the same effect on the flow. Increasing L_1 (see Massoudi and Johnson [52]) and/or L_2 , (see Fig 5) and thus the significance of lift, causes the granular solid to move away from the inner wall and collect towards the outer wall. Again, the changes in the volume fraction due to changing the lift coefficients have relatively little effect on the velocity profiles.

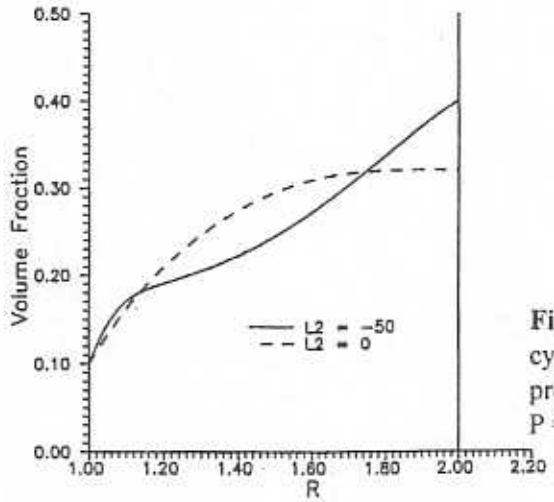

Figure 5. Couette Flow with rotating inner cylinder: Effect of lift on volume fraction profiles; $D_2 = 5$; $D_1 = 10$, $B = 4$, $E = 4$, $L_1 = 0$, $P = -1$, $N = 0.3019$.

Figures 6 through 10 show results analogous to those discussed above for a stationary inner cylinder and a rotating outer cylinder. Increasing D_2 results in an increase in fluid velocity and a corresponding decrease in solid velocity (see

Fig 6). Physically, an increase in the value of D_2 corresponds to either an increase in the drag coefficient or a decrease in the non-dimensional number B_3 (related to

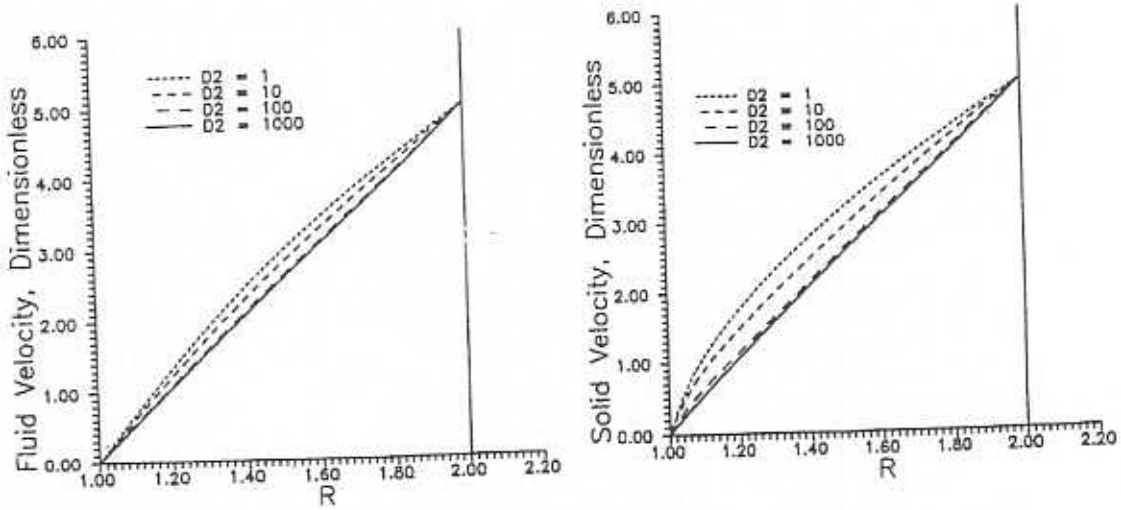

Figure 6. Couette Flow with rotating outer cylinder: Effect of D_2 on fluid and solid velocity profiles; $D_1 = 10$, $B = 4$, $E = 4$, $L_1 = L_2 = 0$, $P = -1$, $N = 0.4$.

the viscosity of the granular materials). Figure 7 shows the fluid and solid velocities for two different values of D_2. Note again the reversal of the effects for rotating outer cylinder as compared to the rotating inner cylinder. Figure 7 also shows the solid leading the fluid when the outer cylinder is rotating; Figure 2 shows the fluid leading the solid when the inner cylinder is rotating.

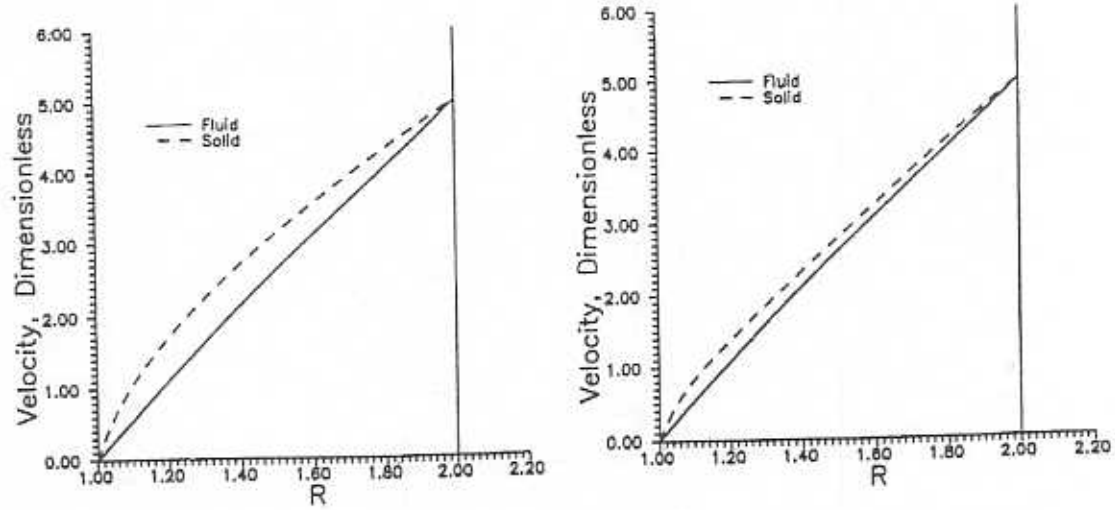

Figure 7. Couette Flow with rotating outer cylinder: Comparison of fluid and solid velocity profiles; a) $D_2 = 1$ (left), and (b) $D_2 = 10$ (right); $D_1 = 1$, $B = 4$, $E = 4$, $L_1 = L_2 = 0$, $P = -1$, $N = 0.4$.

Figure 8 shows plots of volume fraction versus r for some representative values of B with (L_1=L_2=0, and E=4). Note that an integral condition (Eqn 36) is used and the areas under each of the curves are the same, although the boundary condition at the outer tube wall is different for each curve. Though B and E appear to cause the same relative changes in the volume fraction curves regardless of which cylinder is rotating, the overall shape of the curves are different in Figure 8 versus Figure 3.

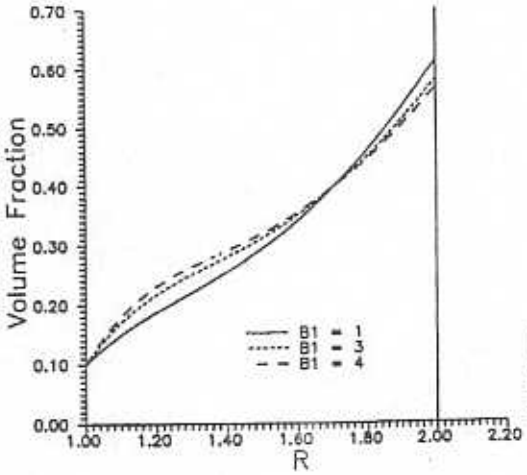

Figure 8. Couette Flow with rotating outer cylinder: Effect of B on volume fraction profiles; D_2 =1; D_1 = 10, E = 4, L_1 = L_2 = 0, P = -1, N = 0.4.

Figure 9 shows the effect of P on velocity profiles. A larger value of P results in a greater portion of the granular solid collecting near the outer cylinder wall (not shown here, see Massoudi and Johnson [52]). Physically, an increase in the magnitude of P corresponds to an increase in ρ_s, the density of the granular solid. Note in Figure 9 that the change in solid distribution has relatively little effect on the velocity profiles. The change in volume fraction profile is more pronounced when the outer cylinder is rotating than when the inner cylinder is rotating (Figure 4).

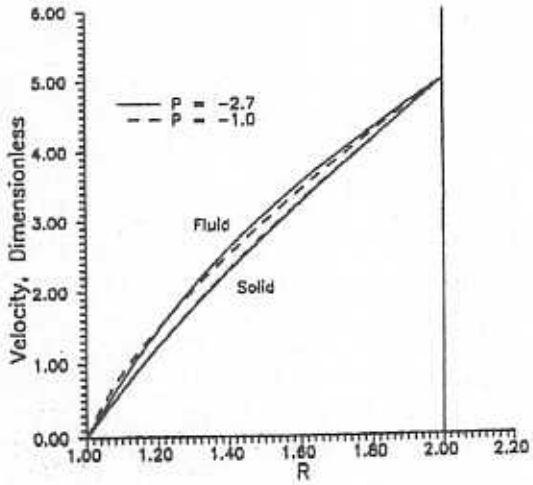

Figure 9. Couette Flow with rotating outer cylinder: Effect of P on velocity profiles; D_2 = 5; D_1 = 10, B = 4, E = 4, L_1 = L_2 = 0, N = 0.2955.

Figure 10 shows a representative result from a parametric study of the effects of 'spin' lift on the volume fraction. Spin lift and slip-shear lift appear to have essentially the same effect on the flow. Increasing L_1 and/or L_2, and thus the significance of lift, causes the granular solid to move away from the inner wall and collect towards the outer wall. This effect appears much less significant for a rotating outer cylinder than for a rotating inner cylinder. Again the changes in volume fraction due to changes in the lift coefficient have relatively little effect on the velocity profiles.

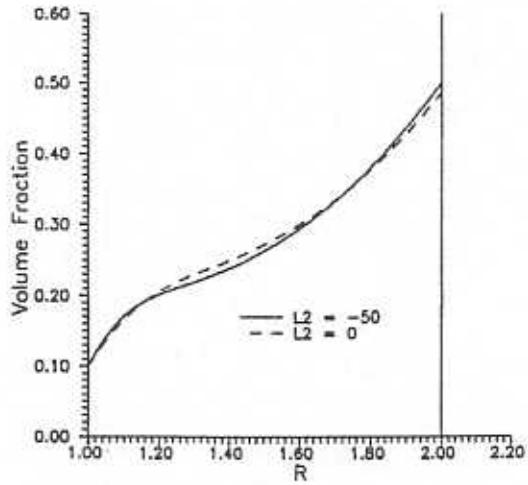

Figure 10. Couette Flow with rotating outer cylinder: Effect of lift on volume fraction profiles; $D_2 = 5$; $D_1 = 10$, $B = 4$, $E = 4$, $L_1 = 0$, $P = -1$, $N = 0.3412$.

The dimensionless traction on the outer cylinder is given by:

$$\mathbf{t} = \mathbf{T_m}.\mathbf{n} = \begin{pmatrix} P(1-\nu) - B_0\nu - (B_1 + B_4)(1+\nu+\nu^2)(\nu')^2 \\ -\frac{1}{Re}(1-\nu)(V' - \frac{1}{r}V) - \frac{1}{2}B_3(\nu+\nu^2)(U' - \frac{1}{r}U) \\ 0 \end{pmatrix} \tag{37}$$

Evaluated at $r=r_0$. The traction on the inner cylinder is given by the same expression, but acting in the opposite direction and evaluated at $r=1$. The torque per unit length acing on the outer cylinder is given by:

$$T = 2\pi r_0{}^2 T_{22} = -\frac{2\pi r_0}{Re}(1-\nu)(r_0 V' - V) - \pi r_0 B_3(\nu+\nu^2)(r_0 U' - U) \tag{38}$$

And the torque per unit length acting on the inner cylinder is given by:

$$T = 2\pi T_{22} = \frac{2\pi}{Re}(1-\nu)(V'-V) - \pi B_3(\nu+\nu^2)(U'-U) \tag{39}$$

These are some of the practical quantities of interest. In summary we can see that similar patterns are observed for the two cases of the inner cylinder or outer cylinder rotating.

SUMMARY AND CONCLUSIONS

Inner cylinder rotating:

. Increase in D_1 , i.e., increase in C_2 (drag) or Re (Reynolds number)
  . decrease in dimensionless fluid velocity
  . decrease in dimensionless solid velocity
  . no change in particle distribution

. Increase in D_2 , i.e., increase in C_2 (drag) or decrease in B_3 (solid viscosity)
  . increase in dimensionless fluid velocity
  . decrease in dimensionless solid velocity
  . no change in particle distribution

. Increase in B_1 results in a decrease in ν maximum (N constant)

. Increase in E results in a decrease in ν maximum (N constant)

. Increase in P results in a decrease in ν maximum (N constant)

. Increase in L_1 or L_2
  . particles collect towards outer cylinder
  . fluid velocity increases slightly where volume fraction decreases
  . fluid velocity decreases slightly where volume fraction increases

Outer cylinder rotating

. Increase in D_1 , i.e., increase in C_2 (drag) or Re (Reynolds number)
  . increase in dimensionless fluid velocity
  . increase in dimensionless solid velocity
  . no change in particle distribution

. Increase in D_2 , i.e., increase in C_2 (drag) or decrease in B_3 (solid viscosity)
  . decrease in dimensionless fluid velocity
  . decrease in dimensionless solid velocity
  . no change in particle distribution

. Increase in B_1 results in a decrease in ν maximum (N constant)

. Increase in E results in a decrease in ν maximum (N constant)
. Increase in P results in a decrease in ν maximum (N constant)
. Increase in L_1 or L_2
-particles collect towards outer cylinder
-fluid velocity increases slightly where volume fraction decreases
-fluid velocity decreases slightly where volume fraction increases

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