ENERGY ANALYSIS OF SOME SINGULAR INITIAL VALUE PROBLEMS

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Introduction

This paper deals with a problem of conservation of energy in a singular initial value problem. These kind of problems appear in the study of radially symmetric elliptic and hyperbolic boundary value problems. The interesting question that arises is how to analyze the oscillation of the solutions, count the number of nodes and estimate their location. In order to conduct the analysis we need to have a very good understanding of the energy of the corresponding solutions and prove results that show how the system contains enough energy to provide oscillation of the solution. We also measure the rate of energy decay in every oscillation that occurs. These results are crucial in solving a number of problems appearing in geometry, engineering, physics, etc.

Defining the Problem

A number of boundary value problems arising in mathematical physics and other applications can be written as a nonlinear elliptic problem of the following type:

(1)

_u(x) + g(u(x)) = p(x), x in _,

u(x) = 0, x in __,

where _ denotes the Laplacian operator, (i.e. the sum of second partial derivatives of u), and _ is a bounded region in R^N. The function g is called the nonlinearity and it is a function from the set of real numbers R to real numbers R.

A problem of central importance is to understand how the properties of a nonlinearity involved in a problem influence the nature of the solutions to the boundary value problem. Because of the wide range of phenomena that occur, these problems must necessarily be undertaken for highly restricted classes of nonlinearities g. A great deal of research has been devoted to the study of the radially symmetric case of the boundary value problem (1), when _ is a ball or radius T (See references [1] -[16]). In the radial case the equation is reduced to a singular ordinary differential equation, and we study the solutions to the following initial value problem

$$u'' + ((N-1)/t)u' + g(u) = p(t), \text{ for } t \text{ in } (0,T)$$

$$u(0) = d, \tag{2}$$

$$u'(0) = 0,$$

where d is a real number. It can be shown (see[6]) using the arguments of the contraction mapping principle that problem (2) has a unique solution u(t,d) continuously depending on the initial condition d. This result is crucial in the sense that it enables us to apply the intermediate value theorem in the phase plane analysis. Since it can be shown that the solutions to (2) that satisfy u(T) = 0 are radially symmetric solutions to the original boundary value problem (1), most of the radially symmetric problems involving elliptic partial differential equation are solved by reducing them to second order ordinary differential equation problems. Therefore, considering the radially symmetric case simplifies the problem a great deal, however the equation obtained is singular and understanding and controlling the singularity presents a serious challenge.

Main Results

In this paper we analyze the equation in (2) focusing on the effect of singularity that appears in the equation (t = 0 in the second term). Our main concern is to eliminate the possibility of energy vanishing due to the singularity. Therefore, we concentrate on showing that the energy remains sufficiently large for solution to oscillate an arbitrary number of times.

In order to analyze energy of the corresponding solutions u(t,d) we define the energy function in the following way:

$$E(u(t,d)) = ((u'(t,d))^2)/2 + G(u(t,d)), \tag{3}$$

where G is the primitive function of g, in other words G(u) = _ g(u)du. We notice that the first term in (3) denotes kinetic energy and the second term denotes potential energy.

Our results show that the energy function remains large enough for the solution to oscillate and, depending on the nature of the nonlinearity, it can oscillate an infinite number of times or only finitely many times.

The results presented below are given for different kind of nonlinearities that occur in application. We start with the superlinear nonlinearity and state the result with the prototypical example of $g(u) = u^3$. Results apply to more general superlinear nonlinearities, however in this paper we consider the following problem:

$$
(4)\qquad \begin{cases} u'' + ((N-1)/t)u' + u^3 = p(t), & \text{for } t \text{ in } (0,T) \\ u(0) = d, \\ u'(0) = 0, \end{cases}
$$

In order to analyze the energy and count the number of oscillations we estimate from below the value of $t^* > 0$ such that the solution to (2) satisfies $v(t^*) = kd$, where k is a constant in-between 0 and 1. Combining this estimate, radial symmetry and the fact that the function is superlinear we show that $E(t)$ tends to infinity as d goes to infinity. Therefore the solution to (4) oscillates arbitrary many times. (See [1], [2]).

We prove a similar result when the nonlinearity is the so called jumping nonlinearity, for example superlinear on one side and constant on the other. The energy analysis is far more demanding and requires a detailed oscillation by oscillation investigation. We also obtain an estimate on the number of nodes involved in the oscillation. (See[3]).

A number of problems in physics such as Yambe's problem lead to a special kind of a boundary value problems where the growth of the nonlinearity reaches the so-called critical exponent. Since most of the standard methods fail in that case a considerable amount of research today is focusing on solving problems involving critical exponents. Here we consider the following prototype:

$$
(5)\qquad \begin{cases} u'' + ((N-1)/t)u' + u + u|u|^s = 0, & \text{for } t \text{ in } (0,T) \\ u(0) = d, \\ u'(0) = 0, \end{cases}
$$

where s is defined as $4/(N-2)$, hence being in direct relationship with N, which denotes the dimension of the space. The energy analysis of the problem (5) provides very detailed information on the location of the nodes. We remark that the Bessel-like equation in (5) oscillates an arbitrary number of times with the nodes being equally spaced. More precisely, if x' and x" denote two consecutive nodes of u then x" - x' = _. Also an interesting result is the connection of the energy estimate with the location of the nodes. We obtain the following: $E(x') < M/x'$, for every node x' and some constant M. Furthermore the decay of energy is measured and given by the following result: If $x' < x''$ are two consecutive zeroes of u (t,d), then

$$E(x'') < (E(x')/N) \{ (x'/x'')^N + (1/N) (1 - (x'/x'')^N)\}.$$

This result indicating how the energy is changing with every oscillation is one of the main ingredients in solving the problem regarding the multiplicity of the solutions to (5) as well as the nature of oscillations. (See [4]-[10]).

Conclusion

A number of applications are modeled by nonlinear boundary value problems and their solutions are radially symmetric functions. That proves to be of great importance since it allows methods of ordinary differential equations to be applied. However, the fact that the equation becomes singular in the ordinary differential equations case poses a problem and requires a very detailed energy analysis to be conducted. Results show that the energy of the solutions to the corresponding initial value problem remains large enough to allow the solution to oscillate an arbitrary many times. Certain variations that occur are related to the dimension of the space N.

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Appendix

Research Project Questions:

The research work presented in this paper is part of an ongoing project focusing on solving elliptic boundary value problems. The project, which is more than a decade old, has generated more than 10 papers, some listed here [1]-[10] and numerous conference presentations as well as two special sessions at the conferences which have attracted many researchers working in this field. The project has been mainly supported through a number of grants from National Science Foundation. Those grants were in the area of applied mathematics and the PI was A. Castro of University of North Texas. Since 1997 the project is supported by another NSF grant and A. Kurepa is the principal investigator. The grant is based at North Carolina A&T State University and it is ending in 1999. We plan to apply for another grant for the year 2000-2003. Hugh Weithers, who is a graduate student in the applied mathematics program at North Carolina A&T State University is working on some problems proposed by the grant and has together with his mentor A. Kurepa obtained new, publishable results. Those results are being presented this spring at three conferences and the paper has been prepared for publication. H. Weithers will complete his M.S. Degree in applied mathematics in May of 1999.

Programmatic Questions:

The Mathematics Department at North Carolina A&T State University offers undergraduate degrees in Mathematics, Applied Mathematics and Mathematics Education and graduate degrees in Applied Mathematics and Mathematics Education.

The Master of Science program in Applied Mathematics at North Carolina A&T State University is a demanding and intellectually rigorous program. Students are thoroughly evaluated as to their backgrounds and any weaknesses are addressed in the carefully tailored program of study worked out by the student and their faculty committee. Graduate applied mathematics faculty members at North Carolina A&T are active and dedicated scholars, with an enthusiastic interest in teaching and ongoing, significant research programs. In addition, the faculty displays a refreshing and challenging diversity with numerous minority and female faculty members serving as role models for graduate and undergraduate students. All graduate faculty teach on both the graduate and undergraduate levels. Their ready availability to the students accounts, in part, for the development of many students, who, in a different environment would languish. As part of their project or master thesis graduate students participate in the research and are working under the supervision of their advisor.

In addition to a few departmental teaching fellowships available to students, since 1995 support for graduate students has been forthcoming from U.S. Department of Education through the GAANN grant (Graduate Assistance in Areas of National Need). The principal investigator for the grant is Alexandra Kurepa. Currently the grant supports 8 graduate students, covering the tuition and providing an annual stipend. The grant has enabled both the students and faculty to concentrate on the academic component of the studies and the graduation rate has increased. All of our students in the program have completed their degree in a timely manner and have participated in research projects.

The Mathematics Department at North Carolina A&T State University is at a critical juncture with regards to its graduate program. Since the U.S. Department of Education is phasing out the GAANN program no new grants have been available and the current funding is ending in the summer of 1999. Without outside funding almost all of our students will be forced to look for employment and do their studies part-time. That will drastically reduce their ability to participate in research projects, as well as their overall success rate in completing the degree. Based on our experiences with the GAANN grant we believe that the best way to support students is to support departments and designate them to recruit good students. We are currently looking for a partnership of that nature, and are interested in working with agencies such as U.S. Department of Energy, US EPA, etc. Funding from government agencies will push the Department to a new level that will insure its prominence and validate and bolster the Department's claims for a larger commitment of the State's resources.

The Department of Mathematics is actively presenting its research at various national and state conferences. It also provides a forum for other researchers to visit the department and share their results by giving talks as part of The North Carolina A&T Mathematics Lecture Series. The Series started in 1995 and has been supported by grants from Exxon and US Department of Education. Funding from Raytheon is approved and expected for 1999/2001 The P.I. for the grants is A. Kurepa.

Having long been highly successful at undergraduate education, and having relatively recently (1986) put into place an academically rigorous graduate program in applied mathematics, the Department is poised to make significant contributions to graduate education of underrepresented groups.