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AN ANALYTICAL STUDY OF TRANSIENT BEHAVIOR OF NONHOMOGENEOUS LINEAR AND RADIAL SYSTEMS

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By
Anil K. Ambastha
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TABLE OF CONTENTS

Page

LIST OF TABLES

Page

LIST OF FIGURES

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ABSTRACT

In performing material balance calculations as proposed by Van Everdingen and Hurst (1949), and Hurst (1958), the dimensionless aquifer response to a unit pressure drop or a unit fluid-withdrawal rate is needed. Later, Mueller (1962) used finite-difference techniques to compute dimensionless pressure drop or cumulative influx for nonhomogeneous aquifers.

This study presents new analytical solutions for the dimensionless pressure drop or cumulative influx for nonhomogeneous aquifers whose thickness, permeability-viscosity ratio, or porosity-compressibility vary linearly with distance. These solutions apply to linear flow conditions with finite aquifer size. The inner boundary condition may be either constant rate or constant pressure. The outer boundary may be either closed, or at constant pressure.

Based on this study, a nonhomogeneous aquifer behaves as a homogeneous aquifer at early times. At late times, pseudosteady, exponential depletion, or steady state behavior is observed, depending on the outer boundary condition. But the dimensionless time to late time state is dependent on the type and severity of the property variation. Limiting solutions at early and late times are presented for each case.

Attempts to obtain analytical solutions for radial nonhomogeneous systems with properties varying linearly with radial distance were unsuccessful. However, a discussion of techniques considered appears in this study. The reasons why the techniques failed may be helpful in future attempts to solve the problem.

1. INTRODUCTION

Many hydrocarbon reservoirs are at least partly bounded by water-bearing rocks called aquifers. The aquifer size may appear infinite or finite compared to the reservoirs they adjoin. The outer boundary of a finite aquifer may be either closed, at a constant pressure, or combinations thereof. The aquifer supplies water to the adjoining reservoir in response to a reservoir pressure decline with time because of hydrocarbon production. Past studies have concerned linear, radial, and spherical flow in the aquifers. Several studies of water influx have been described by Craft and Hawkins (1959).

Schilthius (1936) first proposed a steady-state water influx model. A significant advance was made by the Van Everdingen and Hurst (1949) unsteady-state water influx model that used the response of a dimensionless aquifer with time. They considered the aquifer to be homogeneous and infinite in extent. Later, Hurst (1958) presented a simplified solution of a material balance which used the response for linear and radial homogeneous, infinite aquifers. Miller (1962), and Nabor and Barham (1964) presented analytical solutions for linear, homogeneous aquifers. Chatas (1953) presented extended solutions for radial, homogeneous aquifers. Chatas (1953), and Nabor and Barham (1964) considered both constant rate and constant pressure inner boundary conditions. They also considered the outer boundary to be infinite, closed or at constant pressure. Mueller (1962) studied the problem of the transient response of nonhomogeneous aquifers, and presented finite-difference solutions for finite linear and radial aquifers. He considered linear thickness, permeability or ϕc_t variation with distance from the producing boundary.

Horne and Temeng (1982), and Gerard and Horne (1982) considered the effects of pinch-out boundaries on pressure transient response of a well producing from a formation of nonconstant thickness. They developed analytical solution to the diffusivity equation in radial coordinates by superposing line source segments in a vertical plane passing through the well. Details of the solution technique and the results can be found in the report by Temeng (1981). Thus, analytical solutions for the response from nonhomogeneous systems have been presented only for limited cases so far. This report seeks to fill that gap. Analytical solutions for finite, linear aquifers may be found for linear thickness, permeability, or ϕc_t variation with distance in Section 2, and various appendices. However, only approximate analytical solutions were found for radial nonhomogeneous systems, and are discussed in Section 3. Section 4 presents the results for linear aquifers. Conclusions are presented in Section 5.

2. MATHEMATICAL MODEL FOR LINEAR AQUIFERS

A partial differential equation describing the flow of a slightly compressible fluid in porous media is:

$$\nabla \cdot \left[\frac{kh}{\mu} \nabla p \right] = \phi c_t h \frac{\partial p}{\partial t} . \tag{2.1}$$

In writing Eq. 2.1, we assume a horizontal formation, no gravity forces, isotropic properties, a single fluid of small but constant compressibility, etc.

For a linear, one-dimensional medium, Eq. 2.1 can be written as:

$$\frac{\partial}{\partial x} \left[\frac{kh}{\mu} \frac{\partial p}{\partial x} \right] = \phi c_t \, h \frac{\partial p}{\partial t} . \tag{2.2}$$

As indicated by Mueller (1962), kh/μ and $\phi c_t h$ are two groups of variables which can be a function of the space variable. Thickness appears in both groups. So one may study cases wherein either k/μ (or k), h or ϕc_t is a function of the space variable. Let us identify these as different cases:

CASE I --- Linear thickness variation,
CASE II --- Linear permeability, or permeability-viscosity ratio variation, and
CASE III --- Linear porosity-compressibilty product variation.

As introduced by Mueller (1962), the dimensionless parameter, β, is defined to be the ratio of the value of the variable group at the outer boundary to that at the inner boundary. We will often refer to β as "property ratio". For example, in the case of thickness variation:

$$\beta = \frac{h_o}{h_I} , \tag{2.3}$$

where h_o (h_I) is the thickness at the outer (inner) boundary.

A schematic sketch showing linear thickness variation is presented in Fig. 2.1. For all cases, the usual dimensionless quantities have been used, except that all the properties used in these definitions are defined at the inner boundary. In the case of thickness variation:

$$x_D = \frac{x}{L} , \tag{2.4}$$

$$t_D = \frac{kt}{\phi \mu c_t L^2} , \tag{2.5}$$

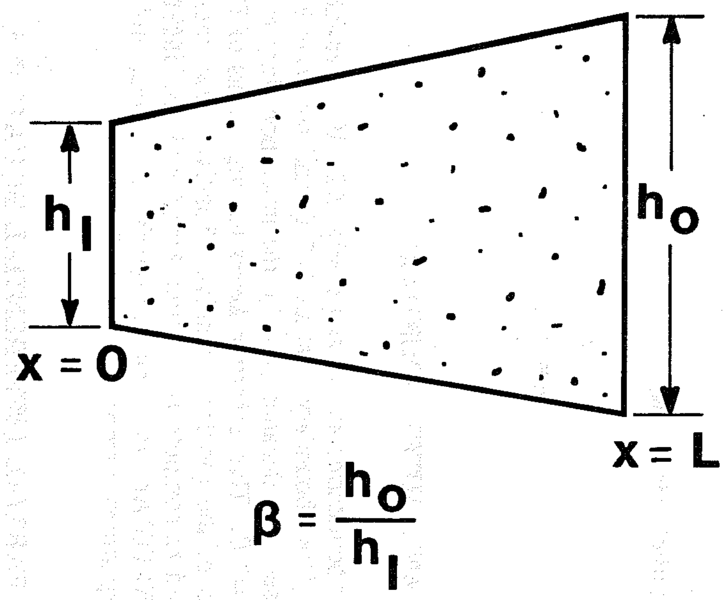


Figure 2.1: Schematic diagram showing linear thickness variation.

$$p_D = \frac{k\, b\, h_I}{q\, \mu\, L} \left[p_i - p(x,t) \right] , \tag{2.6}$$

for a constant-rate inner boundary condition, and

$$p_D = \frac{p(x,t) - p_i}{p_o - p_i} , \tag{2.7}$$

for a constant-pressure inner boundary condition.

In the case of a constant-pressure inner boundary condition, the dimensionless influx rate is:

$$q_D = \frac{q\, \mu\, L}{k\, b\, h_I \left[p_o - p_i \right]} , \tag{2.8}$$

and the dimensionless cumulative influx is:

$$Q_D = \int_0^{t_D} q_D \, dt_D \, . \tag{2.9}$$

2.1. AQUIFER INNER AND OUTER BOUNDARY

For all cases, aquifer boundaries are set as per Fig. 2.2. If the variable property increases with distance ($\beta > 1$), the inner boundary is at x_D ~=~ 0, and the outer boundary is at x_D ~=~ 1. If the property decreases with distance ($\beta < 1$), the inner boundary is at x_D ~=~ 1, and the outer boundary is at x_D ~=~ 0. This definition keeps the arguments of Bessel and Airy functions (which are solutions of the problem) positive. Positive arguments are required for the IMSL library that was used to compute Bessel and Airy functions. IMSL library contains the subroutines written in FORTRAN to compute special functions, and is installed on the Petroleum Engineering Department computer VAX 11/750.

2.2. DIMENSIONLESS PARTIAL DIFFERENTIAL EQUATIONS

Equation 2.2 can be written in dimensionless form for different cases identified earlier using the definitions of dimensionless variables presented in Eqs. 2.3 through 2.7. Dimensionless partial differential equations for different cases are:

CASE I:

For the case of $\beta > 1$,

$$\frac{h}{h_I} = 1 + \alpha\, x_D \, . \tag{2.10}$$

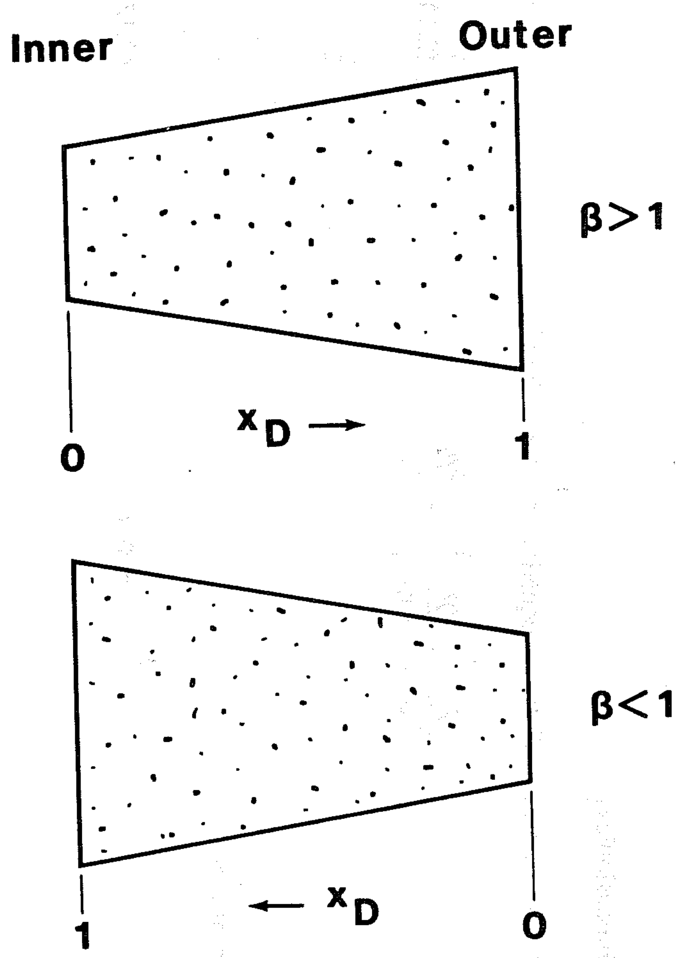


Figure 2.2: Linear aquifer boundaries.

For the case of $\beta < 1$,

$$\frac{h}{h_I} = \beta + \alpha\, x_D \,. \tag{2.11}$$

α is defined as:

$$\alpha = \begin{cases} \beta-1, & \text{for } \beta > 1 \\ 1-\beta, & \text{for } \beta < 1 \end{cases} \tag{2.12}$$

Next let:

$$z = \begin{cases} 1+(\beta-1)\, x_D, & \text{for } \beta > 1 \\ \beta+(1-\beta)\, x_D, & \text{for } \beta < 1 \end{cases} \tag{2.13}$$

Obviously, z is dimensionless.

The dimensionless partial differential equation for case I then is:

$$\frac{\partial}{\partial x_D}\left[z\,\frac{\partial p_D}{\partial x_D}\right] = z\,\frac{\partial p_D}{\partial t_D} \,. \tag{2.14}$$

CASE II:

For the case of $\beta > 1$,

$$\frac{k}{k_I} = 1 + \alpha\, x_D \,. \tag{2.15}$$

For $\beta < 1$,

$$\frac{k}{k_I} = \beta + \alpha\, x_D \,. \tag{2.16}$$

In this case, the dimensionless partial differential equation is:

$$\frac{\partial}{\partial x_D}\left[z\,\frac{\partial p_D}{\partial x_D}\right] = \frac{\partial p_D}{\partial t_D} \,. \tag{2.17}$$

where α and z are given by Eqs. 2.12 and 2.13 respectively.

CASE III:

For the case of $\beta > 1$,

$$\frac{\phi c_t}{(\phi c_t)_I} = 1 + \alpha\, x_D \, . \tag{2.18}$$

For $\beta < 1$,

$$\frac{\phi c_t}{(\phi c_t)_I} = \beta + \alpha\, x_D \, . \tag{2.19}$$

In this case, dimensionless partial differential equation is:

$$\frac{\partial^2 p_D}{\partial x_D^2} = z\, \frac{\partial p_D}{\partial t_D} \, , \tag{2.20}$$

where α and z are given by Eqs. 2.12 and 2.13 respectively.

2.3. DIMENSIONLESS INITIAL AND BOUNDARY CONDITIONS

Initial condition for all cases is:

$$p_D\,(x_D,0) = 0 \, . \tag{2.21}$$

The boundary conditions for $\beta > 1$ follow. The constant-rate inner boundary condition is:

$$\left.\frac{\partial p_D}{\partial x_D}\right]_{x_D=0} = -1 \, , \tag{2.22}$$

whereas the constant-pressure inner boundary condition is:

$$p_D\,(0,\, t_D) = 1 \, . \tag{2.23}$$

The closed outer boundary condition is:

$$\frac{\partial p_D}{\partial x_D}\,(1,t_D) = 0 \, . \tag{2.24}$$

The constant-pressure outer boundary condition is:

$$p_D\,(1,t_D) = 0 \, . \tag{2.25}$$

Corresponding boundary conditions for β <1 follow. The constant-rate inner boundary condition is:

$$\left.\frac{\partial p_D}{\partial x_D}\right]_{x_D=1} = 1 \,. \tag{2.26}$$

The constant-pressure inner boundary condition is:

$$p_D\,(1,\, t_D) = 1 \,. \tag{2.27}$$

The closed outer boundary condition is:

$$\frac{\partial p_D}{\partial x_D}\,(0,t_D) = 0 \,, \tag{2.28}$$

whereas the constant-pressure outer boundary condition is:

$$p_D\,(0,t_D) = 0 \,. \tag{2.29}$$

Appropriate inner and outer boundary conditions are chosen from the preceding set of conditions to evaluate constants in the solution for a particular case.

2.4 SOLUTION METHOD USING LAPLACE TRANSFORM

The Laplace transform of p_D with respect to t_D is defined as:

$$\overline{p}_D\,(x_D,s) = \int_0^{\infty} p_D(x_D,t_D)\, e^{-st_D} dt_D \,. \tag{2.30}$$

The Laplace transform of boundary conditions and the appropriate partial differential equation using the initial condition is taken. The appropriate partial differential equation is thus transformed to the following ordinary differential equations:

CASE I: $$\frac{d}{dx_D}\left[z\,\frac{d\overline{p}_D}{dx_D}\right] - sz\overline{p}_D = 0 \,, \tag{2.31}$$

CASE II: $$\frac{d}{dx_D}\left[z\,\frac{d\overline{p}_D}{dx_D}\right] - s\overline{p}_D = 0 \,, \tag{2.32}$$

CASE III: $$\frac{d^2\overline{p}_D}{dx_D^2} - sz\overline{p}_D = 0 \,. \tag{2.33}$$

Solutions for Cases I and II may be found by comparison with the general Bessel equation (Carslaw and Jaegar, 1959). For Case III, exact solution may be obtained by comparing Eq. 2.33 with the standard Airy equation (Abramowitz and Stegun, 1972). These solutions appear in Appendices A through C. Appendix D presents the solutions for homogeneous aquifer cases using the Laplace transform. Appendix E contains late and early time limiting solutions for different cases.

For constant-pressure inner boundary cases, we are interested in dimensionless cumulative influx behavior. Dimensionless cumulative influx in Laplace space is given as:

$$\overline{Q}_D\,(x_D,s) = \mp \frac{\dfrac{d\overline{p}_D}{dx_D}}{s}\,, \tag{2.34}$$

where the upper sign is for the case of $\beta > 1$, and the lower sign is for $\beta < 1$. The same convention is followed in the appendices. Solutions were inverted from Laplace space to real space using a numerical Laplace transform inverter (Stehfest, 1970).

3. RADIAL NONHOMOGENEOUS SYSTEMS

In this section, we consider the problem of transient response of nonhomogeneous systems with radially-varying properties. Different solution methods are then considered. Some of the methods produced approximate solutions. Limitations and results obtained by applying different methods are discussed.

The general partial differential equation in radial coordinates may be written as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{rkh}{\mu}\frac{\partial p}{\partial r}\right] = \phi c_{th}\frac{\partial p}{\partial t} . \tag{3.1}$$

In writing Eq. 3.1, we assume horizontal flow, negligible gravity effects, isotropic porous medium, a single fluid of small compressibility, and applicability of Darcy's law.

In Eq. 3.1, we can let k (or k/μ), h or ϕc_t be linear functions of radius, r, and study the transient response of such systems. Let us start with permeability as a linear function of radius. This relationship can be expressed as:

$$k = k_I + \frac{(k_o - k_I)}{(r_e - r_w)}(r - r_w) , \tag{3.2}$$

where:

k_I = Permeability at the inner boundary, and

k_o = Permeability at the outer boundary.

The initial condition is:

$$p(r, o) = p_i . \tag{3.3}$$

At the inner boundary, we can have one of the two conditions:

Constant rate:

$$q = -\frac{2\pi k_I h}{\mu}\left[r\frac{\partial p}{\partial r}\right]_{r = r_w} , \tag{3.4}$$

or constant pressure:

$$p\,(r_w,t) = p_w . \tag{3.5}$$

Similarly, at the outer boundary, one of the two conditions hold:

Closed:

$$\frac{\partial p}{\partial r}(r_e,t) = 0 , \tag{3.6}$$

or constant pressure:

$$p(r_e,t) = p_i \tag{3.7}$$

Mixed boundary conditions are not considered here.

3.1. DIMENSIONLESS VARIABLES

For constant rate at the inner boundary, dimensionless pressure drop is defined as:

$$p_D = \frac{2\pi k_I h[p_i - p(r,\ t)]}{q\mu} , \tag{3.8}$$

and for the constant inner boundary pressure case:

$$p_D = \frac{p_i - p(r,\ t)}{p_i - p_w} \tag{3.9}$$

Other dimensionless variables are defined as:

$$r_D = \frac{r}{r_w} , \tag{3.10}$$

$$t_D = \frac{k_I t}{\phi \mu c_t r_w^2} , \tag{3.11}$$

$$\beta = \frac{k_o}{k_I} . \tag{3.12}$$

3.2. PROBLEM IN DIMENSIONLESS FORM

Using the preceding definitions of dimensionless variables, we can write Eq. 3.2 as:

$$\frac{k}{k_I} = 1 + \frac{\beta - 1}{r_{eD} - 1}\,(r_D - 1) , \tag{3.13}$$

where:

$$r_{eD} = \frac{r_e}{r_w} . \tag{3.14}$$

Equation 3.13 can be written as:

$$f(r_D) = \frac{k}{k_I} = a + br_D \ , \tag{3.15}$$

where:

$$a = 1 - \frac{\beta - 1}{r_{eD} - 1} \ , \ \text{and} \tag{3.16}$$

$$b = \frac{\beta - 1}{r_{e_D} - 1} \ . \tag{3.17}$$

Equation 3.1 in dimensionless form becomes:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left[r_D f \frac{\partial p_D}{\partial r_D}\right] = \frac{\partial p_D}{\partial t_D} \ . \tag{3.18}$$

The initial condition in dimensionless form is:

$$p_D\,(r_D, 0) = 0 \ . \tag{3.19}$$

We illustrate all the solution techniques with respect to a constant-rate inner boundary and closed outer boundary, which in dimensionless form are:

$$r_D \frac{\partial p_D}{\partial r_D}\Bigg]_{r_D=1} = -1 \ , \tag{3.20}$$

and:

$$\frac{\partial p_D}{\partial r_D}\Bigg]_{r_D=r_{eD}} = 0 \ . \tag{3.21}$$

Taking the Laplace transform of the partial differential equation, Eq. 3.18, with respect to t_D and using the initial condition, we obtain:

$$\frac{1}{r_D}\frac{d}{dr_D}\left[r_D f \frac{d\overline{p}_D}{dr_D}\right] - s\overline{p}_D = 0 \ , \tag{3.22}$$

which can be written as:

$$\frac{d^2\overline{p}_D}{dr_D^2} + \left[\frac{f'}{f} + \frac{1}{r_D}\right]\frac{d\overline{p}_D}{dr_D} - \frac{s}{f}\,\overline{p}_D = 0 \ , \tag{3.23}$$

where:

$$f' = \frac{df}{dr_D} = b \,. \tag{3.24}$$

Using Eqs. 3.15 and 3.24, Eq. 3.23 can be rewritten as:

$$(ar_D + br_D^2)\frac{d^2\bar{p}_D}{dr_D^2} + (a + 2br_D)\frac{d\bar{p}_D}{dr_D} - r_D s\bar{p}_D = 0 \,. \tag{3.25}$$

Equation 3.25 is a second-order, homogeneous ordinary differential equation with variable coefficients. Boundary conditions in Laplace space are:

$$r_D \frac{d\bar{p}_D}{dr_D}\Bigg]_{r_D=1} = \frac{-1}{s} \,, \tag{3.26}$$

$$\frac{d\bar{p}_D}{dr_D}\Bigg]_{r_D=r_{e_D}} = 0 \,. \tag{3.27}$$

3.3. SOLUTION TECHNIQUES

Before discussing a number of solution techniques in detail, we note that Eq. 3.25 cannot be recast in a form comparable to the general Bessel equation, because of the form of coefficients of $d^2\bar{p}_D \,/\, dr_D^2$ and $d\bar{p}_D \,/\, dr_D$ that involve sum of terms with different powers of r_D.

3.3.1. Similarity Transformation Method

We define a similarity variable η as:

$$\eta = \frac{r_D^2}{4\, t_D} \tag{3.28}$$

This is sometimes known as the Botzmann transformation. A Boltzmann tranformation of this type reduces Eq. 3.18 to:

$$\eta f \frac{d^2 p_D}{d\eta^2} + \left[\frac{d}{d\eta}(\eta f) + \eta\right]\frac{dp_D}{d\eta} = 0 \,, \tag{3.29}$$

which is a first-order, homogeneous ordinary differential equation in $dp_D \,/\, d\eta$. But in order for a similarity transformation method to work, it is also necessary to be able to reduce the number of conditions by one. In our case, Eqs. 3.19 through 3.21 transform to:

$$p_D \mid_{\eta \to \infty} = 0 \,, \tag{3.30}$$

$$8\eta \left. \frac{\partial p_D}{\partial \eta} \right]_{\eta = 1/4t_D} = -1 \,, \tag{3.31}$$

$$\left. \frac{\partial p_D}{\partial r_D} \right]_{\eta = r_{eD}^2 / 4t_D} = 0 \,. \tag{3.32}$$

It is not possible to reduce three conditions to two, because the system is finite. Thus, the Boltzmann transformation does not help for this problem.

3.3.2. Hypergeometric Equation form

Equation 3.25 can be written as:

$$\frac{ar_D + br_D^2}{r_D} \frac{d^2\bar{p}_D}{dr_D^2} + \frac{a + 2br_D}{r_D} \frac{d\bar{p}_D}{dr_D} - s\bar{p}_D = 0 \,. \tag{3.33}$$

Equation 3.33 has three poles at $r_D = \infty$, 0 and $-(a/b)$. Such an equation with three poles can be transformed to a hypergeometric equation for which poles are at ∞, 0 and 1. The hypergeometric equation is written as (Abramowitz and Stegun, 1972):

$$z(1-z)\frac{d^2u}{dz^2} + \left[d - (e + g + 1)z\right]\frac{du}{dz} - egu = 0 \,. \tag{3.34}$$

Equation 3.33 can be transformed by using the substitution:

$$\xi = -\frac{b}{a}\, r_D \tag{3.35}$$

into the form:

$$\xi(1-\xi)\frac{d^2\bar{p}_D}{d\xi^2} + (1 - 2\xi)\frac{d\bar{p}_D}{d\xi} - \frac{as}{b^2}\,\xi\,\bar{p}_D = 0 \,. \tag{3.36}$$

Equation 3.36 has poles at $\xi = \infty$, 0, 1; and is similar to Eq. 3.34, except that the coefficient of $\bar{p}_D$ in Eq. 3.36 is variable. Thus, the problem cannot be reduced to hypergeometric equation form.

3.3.3. Approximate Solution Using Dependent Variable Substitution

Consider Eq. 3.23. Substitute:

$$\bar{p}_D = v\, e^{-1/2 \int \left[\frac{f'}{f} + \frac{1}{r_D}\right] dr_D} = \frac{v}{\sqrt{f r_D}} \tag{3.37}$$

in Eq. 3.23 to obtain:

$$\frac{d^2 v}{dr_D^2} + \alpha(r_D)\, v = 0 \,, \tag{3.38}$$

where:

$$\alpha\,(r_D) = -\frac{s}{f} - \frac{1}{4}\left[\frac{f'}{f} + \frac{1}{r_D}\right] - \frac{1}{2}\frac{d}{dr_D}\left[\frac{f'}{f} + \frac{1}{r_D}\right] . \tag{3.39}$$

If $\alpha(r_D)$ is a slowly-varying function in a region and is negative, then an approximate solution to Eq. 3.38 is:

$$v = (-\alpha)^{-1/4}\,[A \exp\,(\int\sqrt{-\alpha}\; dr_D) + B \exp\,(-\int\sqrt{-\alpha}\; dr_D)] \,. \tag{3.40}$$

If $\alpha(r_D)$ is a slowly-varying function in a region and is positive, then an approximate solution to Eq. 3.38 is:

$$v = (\alpha)^{1/4}\,[C \cos\,(\int\sqrt{\alpha}\; dr_D) + D \sin\,(\int\sqrt{\alpha}\; dr_D)] \,. \tag{3.41}$$

Behavior of $\alpha(r_D)$ is shown for one case in Table 3.1. The function $\alpha(r_D)$ changes sign, and varies significantly in the domain of interest. Therefore, this approximate solution will be of little potential use for this problem.

3.3.4. Approximate Solution Using Independent Variable Substitution

In Eq. 3.22, we use the substitution:

$$\theta = \int \frac{dr_D}{f r_D} = \frac{1}{a} \ln\left[\frac{r_D}{a + b r_D}\right] . \tag{3.42}$$

This substitution results in the transformation of Eq. 3.22 to the form:

$$\frac{d^2 \bar{p}_D}{d\theta^2} - r_D^2 f s\, \bar{p}_D = 0 \,, \tag{3.43}$$

TABLE 3.1: BEHAVIOR OF α (r_D) FOR β =3, r_{eD} =100, AND t_D = 100

r_D	$\alpha(r_D)$
1	0.233
2	5.085e-02
3	1.797e-02
4	6.799e-03
5	1.805e-03
6	-7.960e-04
7	-2.286e-03
8	-3.194e-03
9	-3.771e-03
10	-4.147e-03
15	-4.755e-03
20	-4.696e-03
25	-4.494e-03
30	-4.265e-03
35	-4.040e-03
40	-3.830e-03
45	-3.636e-03
50	-3.459e-03
55	-3.297e-03
60	-3.148e-03
65	-3.012e-03
70	-2.887e-03
75	-2.771e-03
80	-2.665e-03
85	-2.566e-03
90	-2.474e-03
95	-2.388e-03
100	-2.308e-03

which can be simplified to:

$$\frac{d^2\bar{p}_D}{d\theta^2} - \frac{a^3\, e^{2a\theta} s}{(1 - be^{a\theta})^3}\bar{p}_D = 0\,. \tag{3.44}$$

Let:

$$\alpha(\theta) = -\frac{a^3 s\, e^{2a\theta}}{(1 - be^{a\theta})^3}\,. \tag{3.45}$$

Equation 3.44 becomes:

$$\frac{d^2\bar{p}_D}{d\theta^2} + \alpha(\theta)\,\bar{p}_D = 0\,, \tag{3.46}$$

which is similar in form to Eq. 3.38. Again, if $\alpha(\theta)$ is a slowly-varying function and is negative, then an approximate solution is:

$$\bar{p}_D = (-\alpha)^{-1/4}\,[A \exp(\int \sqrt{-\alpha}\, d\theta) + B \exp(\int \sqrt{-\alpha}\, d\theta)]\,, \tag{3.47}$$

If $\alpha(\theta)$ is positive, then an approximate solution is:

$$\bar{p}_D = \alpha^{1/4}\left[C \cos \int \sqrt{\alpha}\, d\theta + D \sin \int \sqrt{\alpha}\, d\theta\right], \tag{3.48}$$

where A, B, C, and D are arbitrary constants in Eqs. 3.47 and 3.48.

Writing approximate solutions in Sections 3.3.3 and 3.3.4 involve solving the corresponding Ricatti equation (Watson, 1944), associated with normal equation such as Eq. 3.38 or 3.46 approximately. Behavior of α (θ) is shown for one case in Table 3.2. Though $\alpha(\theta)$ is always negative in the domain of interest, it varies considerably and thus the assumption of a slowly-varying function is violated. Therefore, we can not use the solution as presented in Eq. 3.47.

3.3.5. Frobenius Method (Series Solution Technique)

Consider Eq. 3.25. The point $r_D = 0$ is a regular singular point for this equation, and so we use the Frobenius method to solve the problem.

The Frobenius method was first used to solve the problem in the domain $(1, r_{eD})$. The solution involved infinite series with terms like $r_{eD}^{\,n-1}$ which grew very large as n became large, even for r_{eD} as small as 20. So the domain of the problem was changed to $(r_{wD}, 1)$ by

TABLE 3.2: BEHAVIOR OF α (θ) FOR β =3, r_{eD} =100, AND t_D = 100

r_D	α(θ)
1	-6.932e-03
2	-2.829e-02
3	-6.490e-02
4	-0.118
5	-0.187
6	-0.275
7	-0.381
8	-0.506
9	-0.652
10	-0.819
15	-2.001
20	-3.837
25	-6.433
30	-9.893
35	-14.323
40	-19.828
45	-26.513
50	-34.482
55	-43.842
60	-54.696
65	-67.150
70	-81.309
75	-97.277
80	-115.161
85	-135.064
90	-157.093
95	-181.351
100	-207.945

the following change of definitions of dimensionless variables and other parameters:

$$r_D = \frac{r}{r_e} , \tag{3.49}$$

$$t_D = \frac{k_I\, t}{\phi \mu c_t r_e^2} , \tag{3.50}$$

$$r_{wD} = \frac{r_w}{r_e} , \tag{3.51}$$

$$f(r_D) = \frac{k}{k_I} = a + b r_D , \tag{3.15}$$

$$a = 1 - \frac{(\beta - 1)\, r_{wD}}{(1 - r_{wD})} , \tag{3.52}$$

$$b = \frac{\beta - 1}{1 - r_{wD}} . \tag{3.53}$$

This change of scale was designated to resolve numerical problems in computation. Using these definitions, we again obtain Eq. 3.25. But inner and outer boundary conditions become:

$$r_D \frac{d\overline{p}}{dr_D} \Bigg]_{r_D = r_{w_D}} = -\frac{1}{s} , \tag{3.54}$$

and:

$$\frac{d\overline{p}_D}{dr_D} \Bigg]_{r_D = 1} = 0 . \tag{3.55}$$

A complete development of the Frobenius method is now illustrated for the preceding problem. Assume the first independent solution to be:

$$Y_1 = \sum_{n=0}^{\infty} a_n\, r_D^{n+u} , \tag{3.56}$$

where u is an undetermined parameter. The two values u can assume are the same, and equal to zero. This can be shown by some algebraic manipulations omitted here. Thus, the first independent solution has the form:

$$Y_1 = \sum_{n=0}^{\infty} a_n\, r_D^{n} , \tag{3.57}$$

and it can be shown that:

$$a_0 = 1 ,$$

$$a_1 = 0 ,$$

$$a_n = \frac{(1 - r_{wD})s\ a_{n-2} - (\beta - 1)\ n\ (n-1)a_{n-1}}{(1 - \beta\ r_{wD})n^2}, \quad \text{for} \quad n \geq 2 . \tag{3.58}$$

Since both values of u are equal to zero, the second linearly independent solution is of the following form (Boyce and DiPrima, 1977):

$$Y_2 = Y_1 \ln (r_D) + \sum_{n=1}^{\infty} b_n\ r_D^n$$

$$= \ln (r_D) + \sum_{n=1}^{\infty} [\ a_n \ln (r_D) + b_n] r_D^n , \tag{3.59}$$

where b_n are undetermined coefficients. Determination of b_n leads to:

$$b_0 = 0$$

$$b_1 = \frac{\beta - 1}{1 - \beta\ r_{w_D}}$$

$$b_n = \frac{s(1 - r_{wD})}{(1 - \beta\ r_{wD})} \left[\frac{n^2\ b_{n-2} - 2na_{n-2}}{n^4} \right]$$

$$- \frac{(\beta - 1)}{1 - \beta\ r_{wD}} \left[\frac{n(a_{n-1} + (n-1)\ b_{n-1}) - n-1\ a_{n-1}}{n^2} \right] \text{for } n \geq 2 \tag{3.60}$$

A general solution to the problem then is:

$$\bar{p}_D = C_1\ Y_1 + C_2\ Y_2 , \tag{3.61}$$

where C_1 and C_2 are arbitrary constants that are determined by using the boundary conditions (Eqs. 3.53 and 3.54).

This solution did not pose a computational convergence problem for the homogeneous system with $\beta = 1$. Spot checks indicated that the solution reproduced the pressure transient response of a homogeneous reservoir case reported in the literature (Aziz and Flock, 1963). Solution for homogeneous reservoir ($\beta = 1$) case using numerical inversion of $\bar{p}_D$ given by Eq. 3.61 is presented in Fig. 3.1.

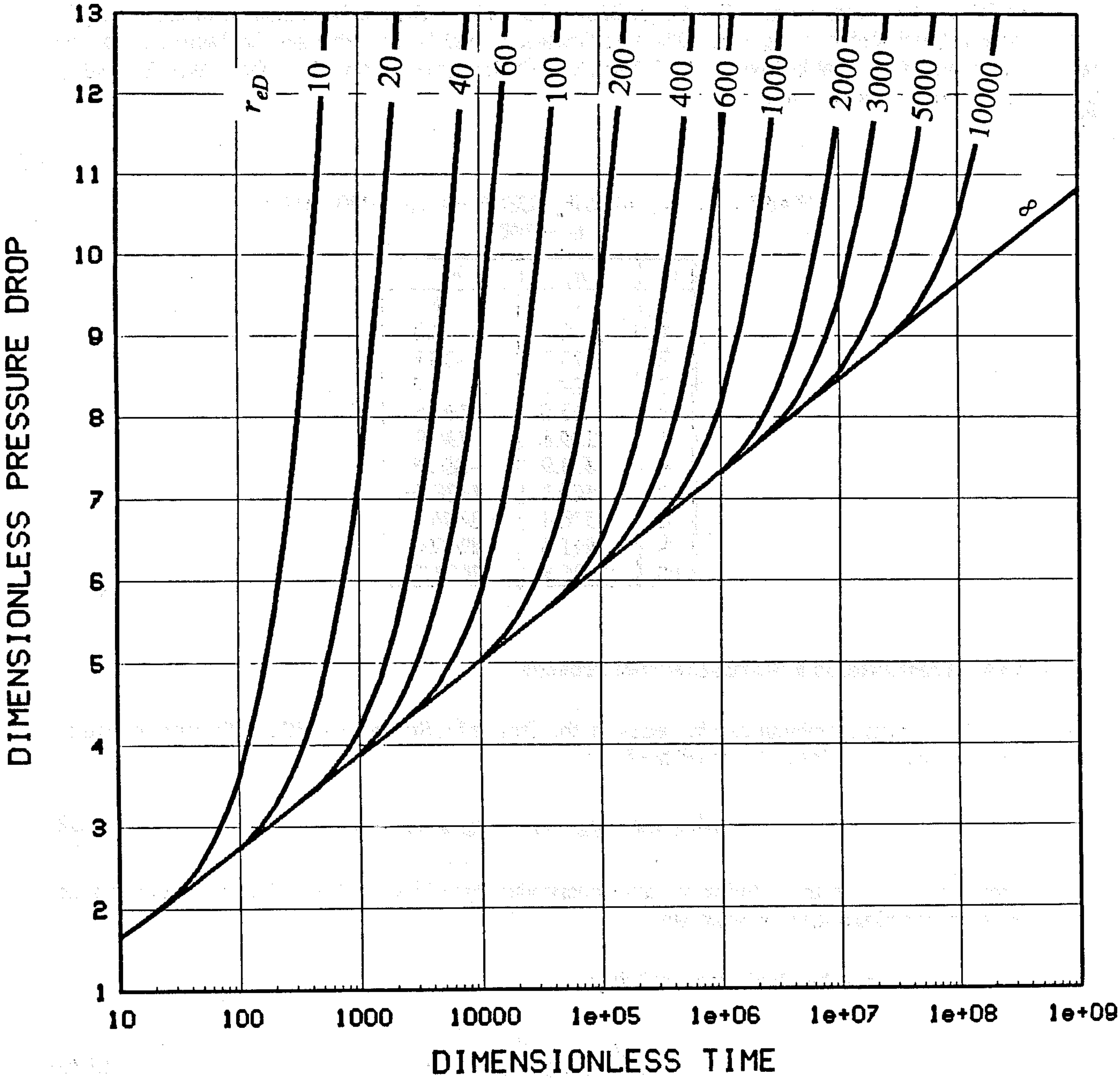


Figure 3.1: Dimensionless pressure behavior for radial homogeneous system (Constant rate inner and closed outer boundary).

The pressure transient response for $\beta = 0.5$ was studied, but convergence was not obtained at early times for several values of r_{eD}. Coefficients Coefficients a_n and b_n are shown in Fig. 3.2 for a typical case. The coefficients a_n and b_n in this case go to zero asymptotically, which explains convergence for the problem. For other values of β, behavior of a_n and b_n for large n precluded convergence. The coefficients a_n and b_n are tabulated in Table 3.3 for one case which is typical behavior. Such oscillatory behavior of a_n and b_n is the reason for lack of convergence in most cases.

TABLE 3.3 a_n AND b_n FOR $\beta = 3$, $r_{eD} = 100$, AND $t_D = 100$

n	a_n	b_n
0	1	0
1	0	2.1
2	17.7	-15.6
3	-24.3	1.1
4	115.8	-106.5
5	-259.8	196.9
6	673.9	-608.5
7	-1566.1	1438.5
8	3570.4	-3403.6
9	-7911.4	7707.4
10	17206.8	-17052.2

3.3.6. Treatment as a Sturm-Liouville Problem

The original problem can be recast in the form of a Sturm-Liouville differential equation, if we assume the solution to be of the form:

$$p_D(r_D, t_D) = p_{D,e}(r_D) + p_{D,h}(r_D, t_D) , \tag{3.62}$$

where $p_{D,e}(r_D)$ is the solution to time-independent problem, and $p_{D,h}(r_D, t_D)$ represents the transient contribution to pressure drop.

$p_{D,e}(r_D)$ may be obtained by solving:

$$\frac{1}{r_D}\frac{d}{dr_D}\left[r_D f \frac{dp_{D,e}}{dr_D}\right] = e , \tag{3.63}$$

where e is a constant. If the boundary conditions are such that the system approaches steady state behavior at late time, the constant e is taken to be zero. This applies to constant-pressure outer boundary situations. If the outer boundary is closed, the system behavior will approach pseudo-steady state at late time, and the constant e represents the slope of dimensionles pressure drop vs. dimensionless time at long time.

Solutions of the time-independent problem with appropriate boundary conditions makes boundary conditions to the transient problem homogeneous. Initial condition to the transient

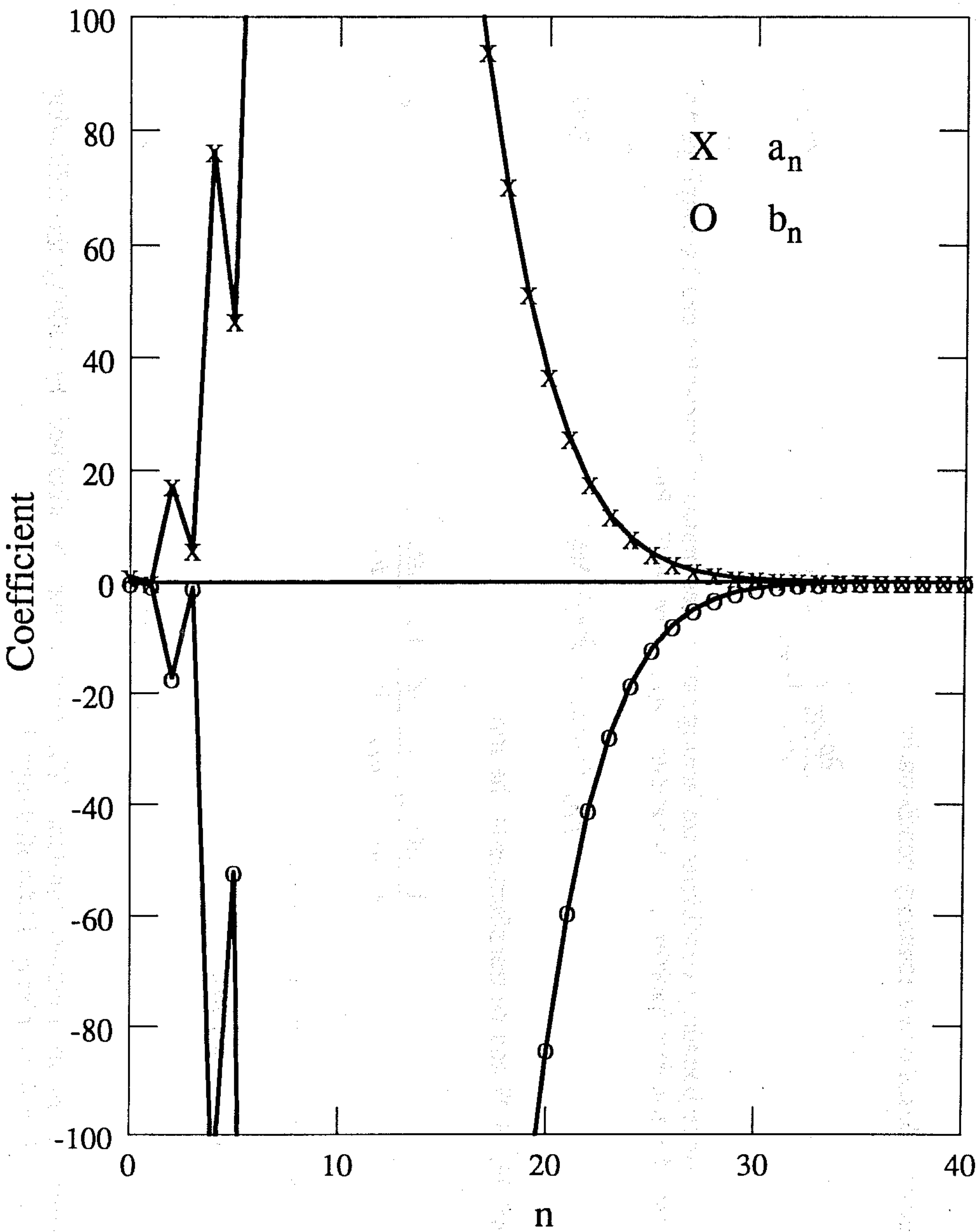


Figure 3.2: Coefficients a_n and b_n for radial system (β =0.5, r_{eD} =100, and t_D =100).

problem will be inhomogeneous. The typical transient problem obtained will resemble:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left[r_D f \frac{\partial p_{D,h}}{\partial r_D}\right] = \frac{\partial p_{D,h}}{\partial t_D} \ . \tag{3.18}$$

The initial condition is:

$$p_{D,h}\,(r_D, 0) = -p_{D,e}(r_D) \ . \tag{3.64}$$

Inner and outer boundary conditions will be homogeneous, e.g., constant rate inner boundary condition is:

$$\left.\frac{\partial p_{D,h}}{\partial r_D}\right]_{r_D=1} = 0 \ , \tag{3.65}$$

and the closed outer boundary condition is:

$$\left.\frac{\partial p_{D,h}}{\partial r_D}\right]_{r_D=r_{e_D}} = 0 \ . \tag{3.66}$$

Since boundary conditions are homogeneous, separation of variables can be employed to solve the transient problem. If we assume $p_{D,h}$ to be of the form:

$$p_{D,h}\,(r_D, t_D) = \hat{\phi}(r_D)\,\hat{\alpha}(t_D) \ , \tag{3.67}$$

then Eq. 3.18 can be transformed to the form:

$$\frac{1}{\hat{\alpha}}\frac{d\hat{\alpha}}{dt_D} = \frac{1}{\hat{\phi} r_D}\frac{d}{dr_D}\left[f r_D \frac{d\hat{\phi}}{dr_D}\right] = -\lambda \ , \tag{3.68}$$

where λ is a separation constant.

It is obvious that:

$$\hat{\alpha} = A\, e^{-\lambda t_D} \tag{3.69}$$

where A is an arbitrary constant. The term $\hat{\phi}$ may be obtained by solving the following Sturm-Liouville ordinary differential equation:

$$\frac{d}{dr_D}\left[f r_D \frac{d\hat{\phi}}{dr_D}\right] + \lambda r_D\,\hat{\phi} = 0 \ . \tag{3.70}$$

The initial condition, Eq. 3.64, determines the constant A in Eq. 3.69. But attempts to obtain an analytical solution to Eq. 3.70 did not succeed. It seems to proceed with a numerical solution to Eq. 3.70, because the primary intention was to solve the problem analytically.

4. RESULTS AND DISCUSSION

Some of the response functions obtained analytically in this work for linear system were first generated by Mueller (1962) numerically. For all cases, capable of solution our results compare very well with his results. Our results are compared with Mueller's solution (1962) for one case in Table 4.1. Homogeneous case results were obtained previously by Miller (1962), Nabor and Barham (1964), and Mueller (1962).

TABLE 4.1 COMPARISON OF DIMENSIONLESS PRESSURE-DROP FOR LINEAR FINITE SYSTEM, CLOSED OUTER BOUNDARY AND CONSTANT RATE INNER BOUNDARY (β=4)

Dimensionless Time	Mueller's solution*	This study
0.001	0.035	0.0343
0.005	0.073	0.0731
0.01	0.1	0.1
0.05	0.198	0.1972
0.1	0.258	0.2573
0.5	0.465	0.4642
1	0.66	0.6648
5	2.3	2.2650
10	4.2	4.2653

Note: * -- Read off Fig. 9 of Mueller (1962)

4.1. THICKNESS VARIATION

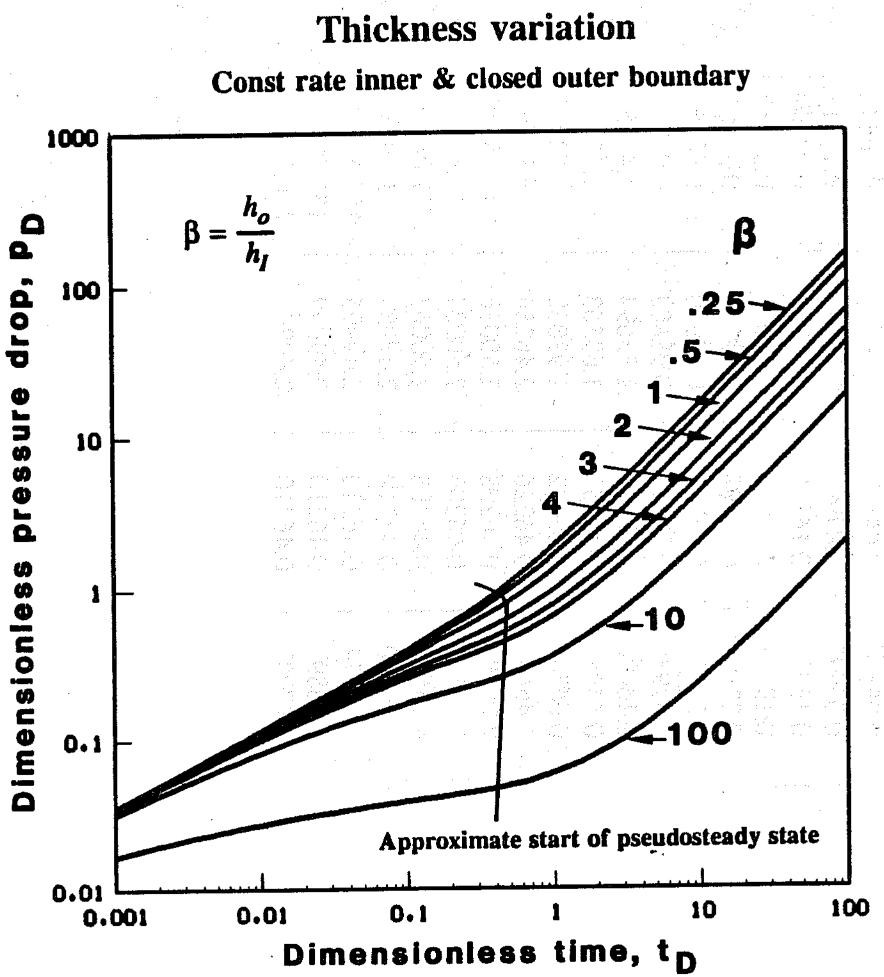
Figures 4.1 through 4.4 present the solutions for various inner and outer boundary conditions where thickness varies linearly with distance. At early times, all results cluster in all cases, except for β = 10 and 100. Actually, results for β = 10 and 100 also approach the β =1 case at dimensionless times lower than shown on the figures. At late times, flow goes to pseudosteady or steady state. The dimensionless time to pseudosteady or steady state is dependent on the "property ratio". The approximate time to pseudosteady or steady state is shown on all figures. The approximate time to pseudosteady or steady state is the time by which the dimensionless pressure drop or cumulative influx is within 2% of those given by corresponding late time solution. When pseudosteady or steady state is attained, all curves become parallel to the curve for a homogeneous aquifer case (β = 1) on a log-log graph. Results from the Stehfest inversion algorithm for β = 4 appear in Table 4.2.

4.2. PERMEABILITY VARIATION

Figures 4.5 through 4.8 present solutions for the cases where permeability varies linearly with distance. Again at early times, all curves (except for β =10 and 100) are clustered together. In the closed outer boundary cases, all curves merge with the homogeneous aquifer case solution at late times (See Figs. 4.5 and 4.7) on log-log graphs. But in the constant pressure outer boundary cases, all curves become parallel to the homogeneous aquifer case, on a log-log graph (See Figs. 4.6 and 4.8).

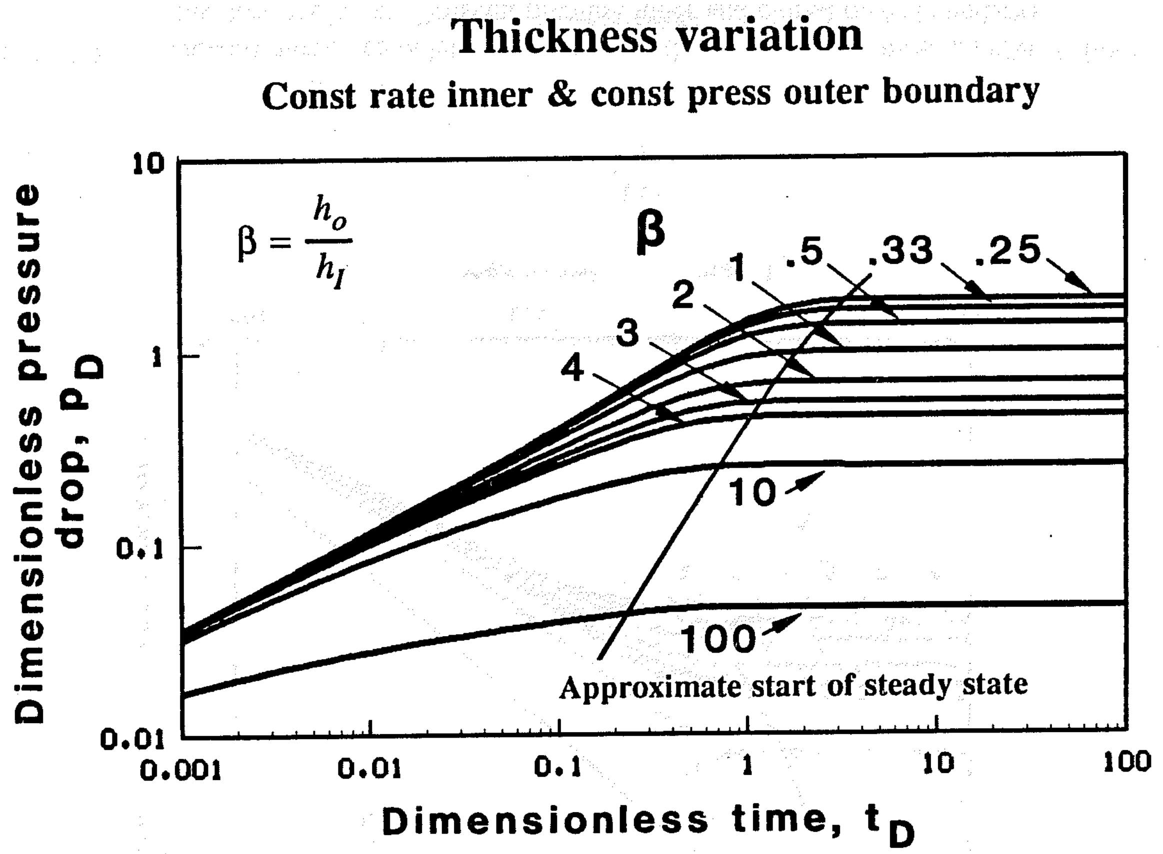
TABLE 4.2: DIMENSIONLESS PRESSURE DROP OR CUMULATIVE INFLUX FOR A LINEAR, FINITE AQUIFER (LINEAR THICKNESS VARIATION, $\beta = 4$)

Dimensionless	Inner boundary	Constant rate	Constant rate	Constant pressure	Constant pressure
time	Outer boundary	Closed	Constant pressure	Closed	Constant pressure
0.001		0.03426	0.03426	0.03716	0.03716
0.003		0.05768	0.05768	0.06618	0.06618
0.005		0.07308	0.07308	0.08703	0.08703
0.007		0.08519	0.08519	0.10448	0.10448
0.01		0.09997	0.09997	0.12713	0.12713
0.03		0.16035	0.16035	0.23710	0.23709
0.05		0.19725	0.19725	0.32055	0.32053
0.07		0.22494	0.22488	0.39278	0.39285
0.1		0.25731	0.25717	0.48930	0.48959
0.3		0.37938	0.37313	0.98808	0.99888
0.5		0.46420	0.42177	1.35694	1.44359
0.7		0.54480	0.44352	1.63563	1.87851
1.		0.66481	0.45632	1.93038	2.52812
3.		1.46482	0.46226	2.46254	6.85664
5.		2.26496	0.46208	2.50036	11.18540
7.		3.06508	0.46206	2.50204	15.51416
10.		4.26527	0.46208	2.50063	22.00729
30.		12.26650	0.46210	2.49984	65.29482
50.		20.26774	0.46210	2.49995	108.58233
70.		28.26897	0.46210	2.49999	151.86985
100.		40.27082	0.46210	2.49999	216.80113



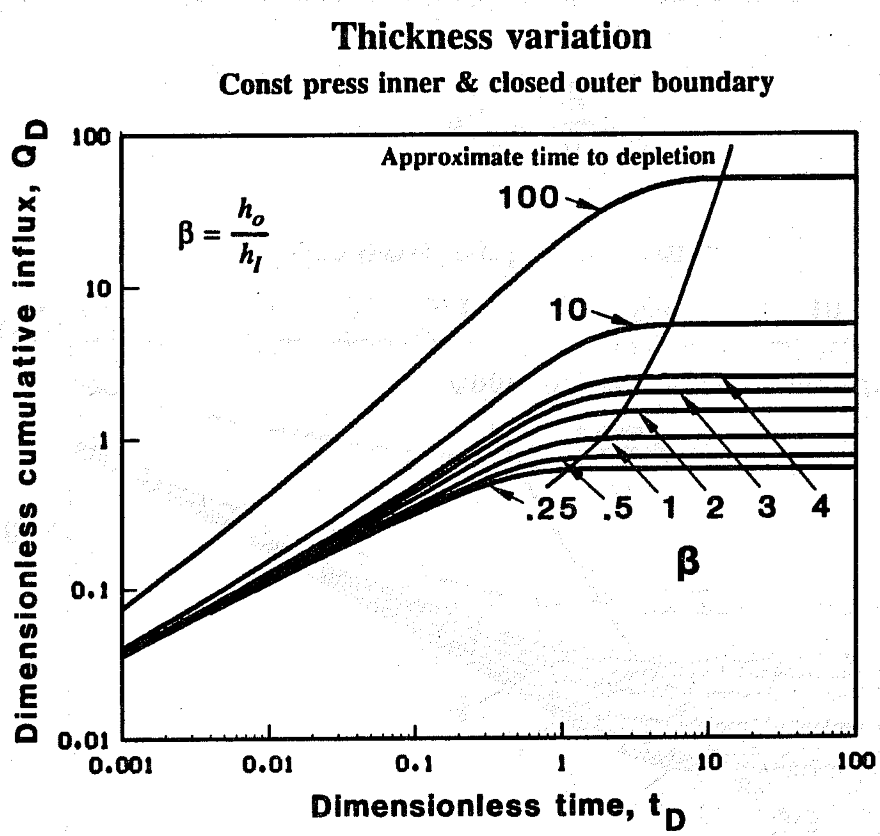
$$p_D = \frac{1}{(\beta+1)}\left[2t_D - \left[\frac{\beta}{\beta-1}\right]^2 [0.5 - ln(\alpha)]\right]$$

Figure 4.1: Dimensionless pressure behavior for linear, finite aquifer (Linear thickness variation, Constant rate inner and closed outer boundary).



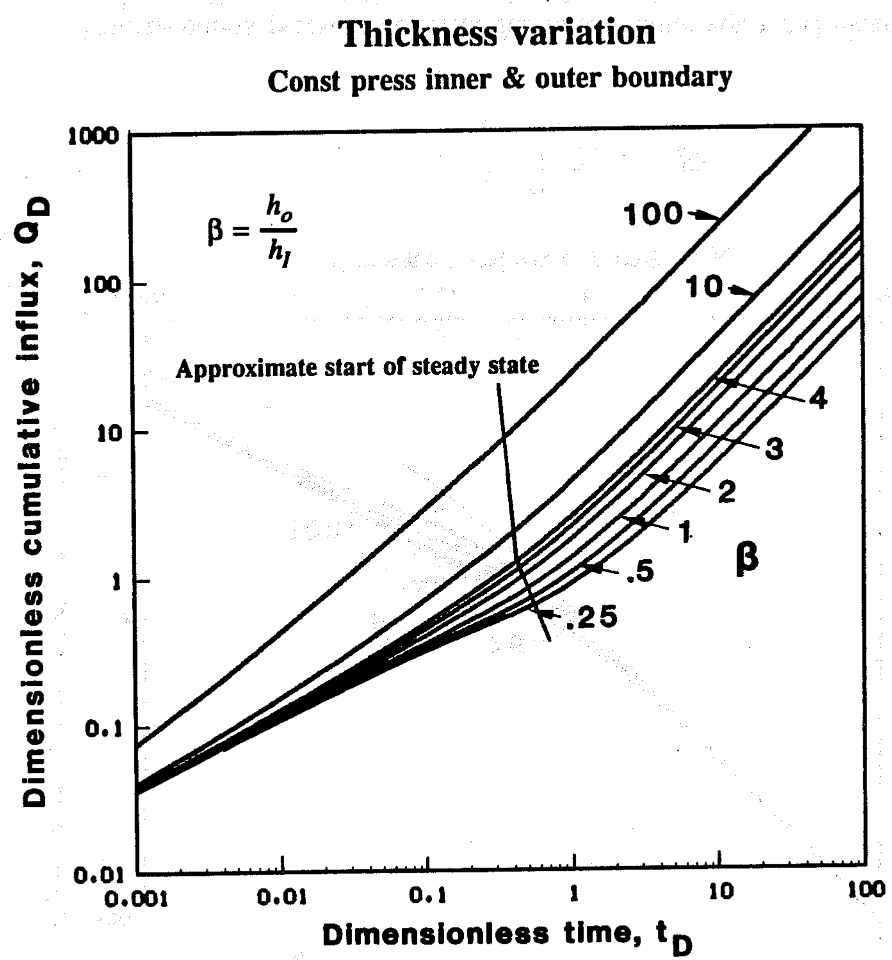
$$p_D = \frac{\ln(\beta)}{\beta - 1}$$

Figure 4.2: Dimensionless pressure behavior for linear, finite aquifer (Linear thickness variation, Constant rate inner and constant pressure outer boundary).



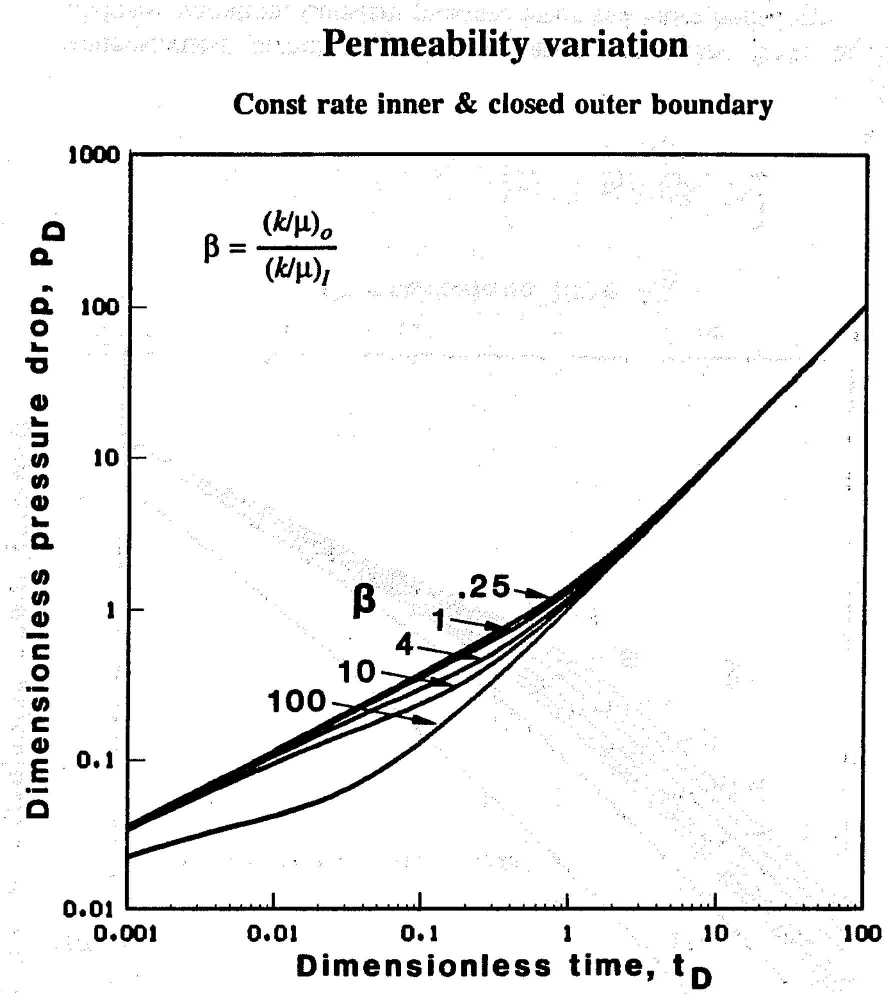
$$Q_D = \frac{\beta+1}{2}$$

Figure 4.3: Dimensionless cumulative influx behavior for linear, finite aquifer (Linear thickness variation, Constant pressure inner and closed outer boundary).



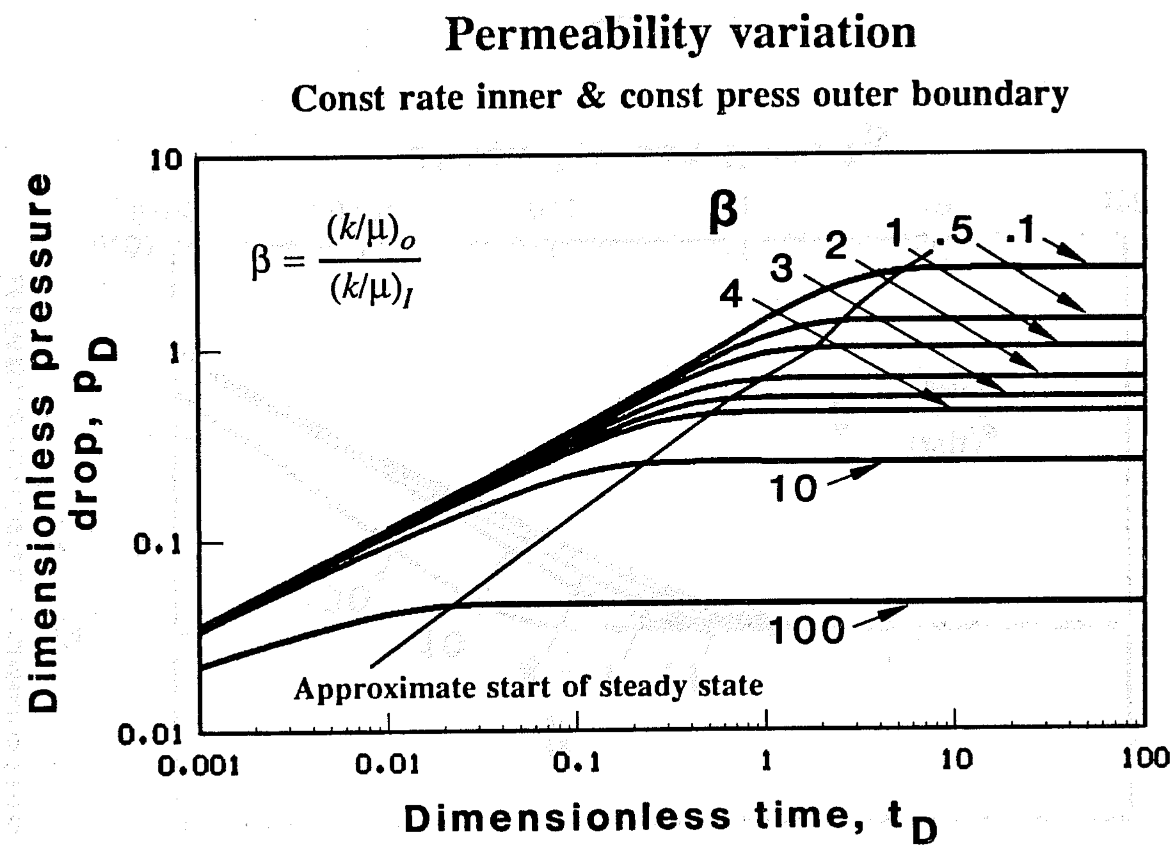
$$Q_D = \pm \frac{\alpha}{\ln(\beta)}\, t_D \pm \left[\frac{-1+2\ln(2\alpha/\beta)-2\gamma}{4\alpha\, ln(\beta)}\right]$$

Figure 4.4: Dimensionless cumulative influx behavior for linear, finite aquifer (Linear thickness variation, Constant pressure inner and outer boundary).



$$p_D = t_D + \frac{\beta}{\alpha^2}\left[1+2ln(\alpha)-2\gamma\right]$$

Figure 4.5: Dimensionless pressure behavior for linear, finite aquifer (Linear permeability variation, Constant rate inner and closed outer boundary).



$$p_D = \frac{ln(\beta)}{\beta - 1}$$

Figure 4.6: Dimensionless pressure behavior for linear, finite aquifer (Linear permeability variation, Constant rate inner and constant pressure outer boundary).

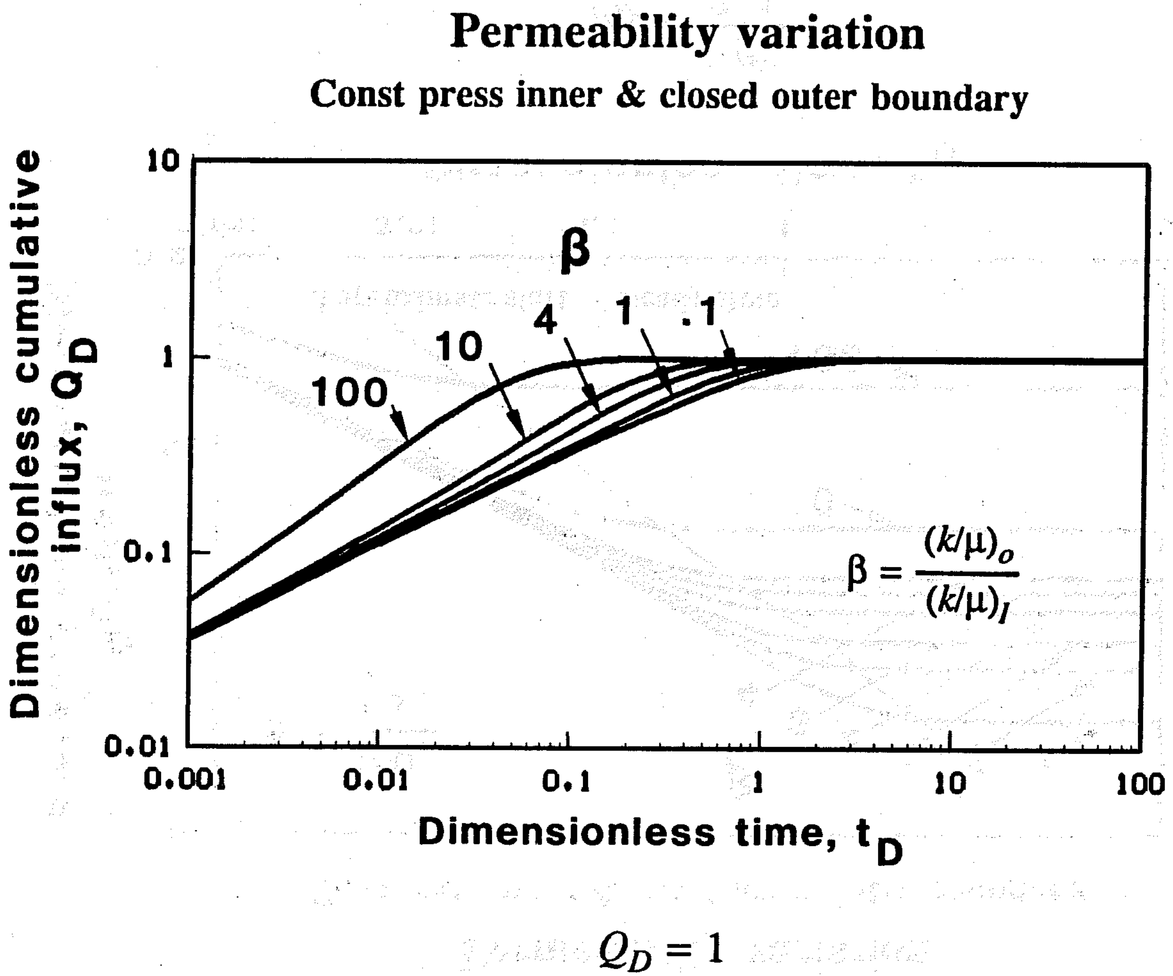
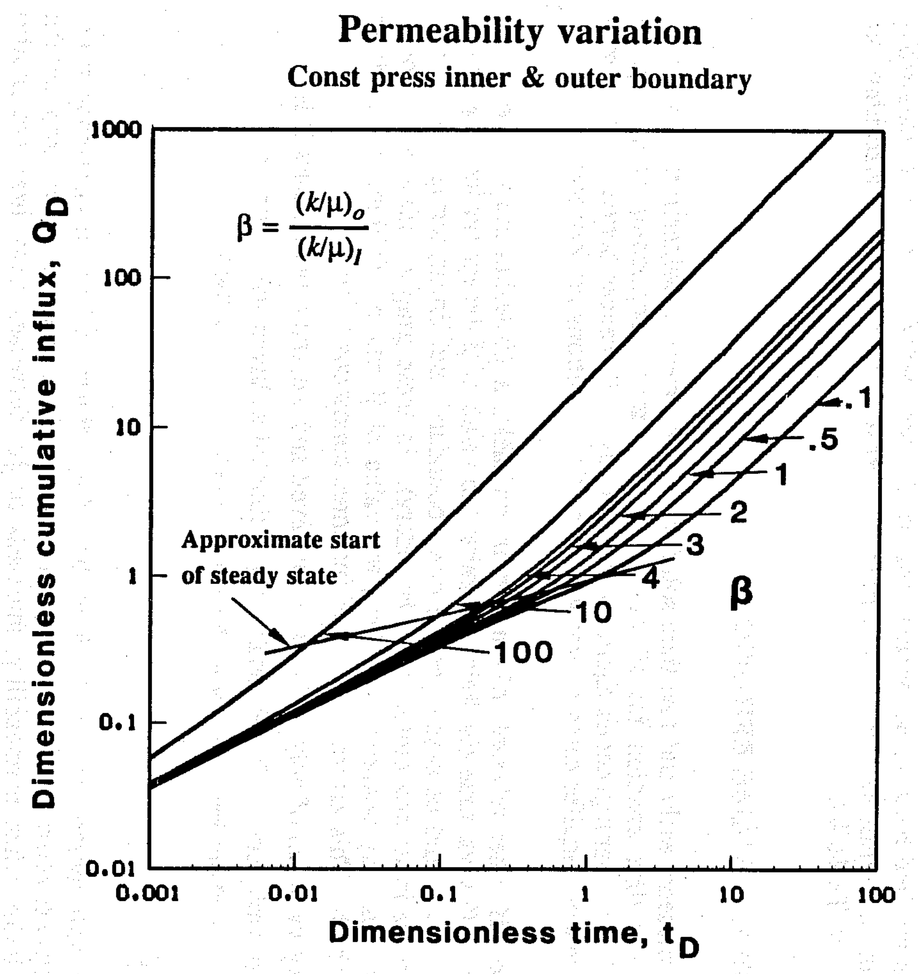


Figure 4.7: Dimensionless cumulative influx behavior for linear, finite aquifer (Linear permeability variation, Constant pressure inner and closed outer boundary).



$$Q_D = \pm \frac{\alpha}{ln(\beta)}\, t_D \pm \frac{\left[1-2\ \ln(\sqrt{\beta}/\alpha)\ -2\gamma\right]}{\alpha\ ln(\beta)}$$

Figure 4.8: Dimensionless cumulative influx behavior for linear, finite aquifer (Linear permeability variation, Constant pressure inner and outer boundary).

As also observed by Mueller (1962), at intermediate times, results for $\beta < 1$ are closer to the homogeneous aquifer case than are results for $\beta > 1$, for the same proportion of increase or decrease in permeability (See Figs. 4.5 through 4.8). This observation is also true for the cases of thickness or porosity-compressibility variation. Results from the Stehfest inversion algorithm for $\beta = 4$ are presented in Table 4.3.

4.3. POROSITY-COMPRESSIBILITY VARIATION

Referring to Figs. 4.9 through 4.12, behavior approaches pseudosteady or steady state at different dimensionless times depending on "property ratio". For closed outer boundary cases, the response functions become parallel to the homogeneous aquifer case on a log-log graph, and for the constant pressure outer boundary cases, response functions merge with the homogeneous aquifer case. This is opposite the observation in the case of permeability variation. These observations resulted because ϕc_t appears only in the accumulation term, and permeability appears only in the diffusive term of the problem. Results from Stehfest algorithm for $\beta = 4$ are presented in Table 4.4.

4.4. EARLY AND LATE TIME BEHAVIOR

Appendix C presents early and late time solutions for the cases considered in this study. Equation C-13 (early time solution for all cases) indicates that a nonhomogeneous aquifer behaves as an equivalent homogeneous aquifer at early times. Differences between homogeneous and nonhomogeneous aquifer cases appear when late time solutions are inspected.

In most cases, if other relevant information is available, a Cartesian graph of pressure drop or cumulative influx vs. time can help determine the value of β except for linear ϕc_t variation (constant-rate inner and constant-pressure outer boundary) and linear permeability variation (constant-pressure inner and closed outer boundary). In these graphs, either slope, intercept, or both, contains only unknown β. Thus, the value of β can be obtained using either slope, intercept or both. This assumes the following three conditions to hold:

1. Property variation of only one class is involved and we know which class it is.
2. Property variation can be reasonably approximated as linear with distance.
3. Knowledge about inner and outer boundary conditions is available.

4.5. DISCUSSION

In all cases with constant rate at the inner boundary, dimensionless pressure drop functions for $\beta < 1$ remain above those for $\beta > 1$. That is, we incur more pressure drop for the same cumulative influx when permeability, thickness or ϕc_t decreases away from the aquifer inner boundary. In all cases with constant pressure at the inner boundary, dimensionless cumulative influx functions for $\beta < 1$ remain below those for $\beta > 1$. That is, for the same pressure drop, cumulative influx is greater if permeability, thickness or ϕc_t increases away from the aquifer inner boundary. A summary of intermediate and late time behavior for different cases appears in Tables 4.5 through 4.7.

Mixed inner and outer boundary conditions were not considered herein. It is not difficult to develop solutions for such cases following the development in this study, if necessary. Also we have restricted consideration to linear variation of the property. Solutions may also be obtained analytically for other types of variation functions, e.g, power law variation (Carslaw and Jaegar, 1959).

TABLE 4.3: DIMENSIONLESS PRESSURE DROP OR CUMULATIVE INFLUX FOR A LINEAR, FINITE AQUIFER (LINEAR PERMEABILITY VARIATION, β = 4)

Dimensionless time	Inner boundary / Outer boundary	Constant rate / Closed	Constant rate / Constant pressure	Constant pressure / Closed	Constant pressure / Constant pressure
0.001		0.03495	0.03495	0.03643	0.03643
0.003		0.05965	0.05965	0.06402	0.06402
0.005		0.07625	0.07625	0.08347	0.08347
0.007		0.08950	0.08950	0.09954	0.09954
0.01		0.10592	0.10592	0.12015	0.12015
0.03		0.17578	0.17574	0.21698	0.21702
0.05		0.22073	0.22058	0.28774	0.28796
0.07		0.25549	0.25536	0.34772	0.34800
0.1		0.29777	0.29681	0.42590	0.42723
0.3		0.51045	0.42280	0.75110	0.87571
0.5		0.71055	0.45241	0.89052	1.30874
0.7		0.91051	0.45995	0.95111	1.74153
1.		1.21054	0.46224	0.98561	2.39081
3.		3.21086	0.46206	1.00040	6.71958
5.		5.21117	0.46207	0.99992	11.04833
7.		7.21148	0.46208	0.99990	15.37708
10.		10.21194	0.46209	0.99994	21.87021
30.		30.21502	0.46210	1.00000	65.15773
50.		50.21810	0.46210	1.00000	108.44525
70.		70.22118	0.46210	1.00000	151.73277
100.		100.22580	0.46210	1.00000	216.66405

TABLE 4.4: DIMENSIONLESS PRESSURE DROP OR CUMULATIVE INFLUX FOR A LINEAR, FINITE AQUIFER (LINEAR POROSITY-COMPRESSIBILITY VARIATION, $\beta = 4$)

Dimensionless	Inner boundary	Constant rate	Constant rate	Constant pressure	Constant pressure
time	Outer boundary	Closed	Constant pressure	Closed	Constant pressure
0.003		0.05962	0.05962	0.06406	0.06406
0.005		0.07618	0.07618	0.08354	0.08354
0.007		0.08940	0.08940	0.09966	0.09966
0.01		0.10650	0.10650	0.11951	0.11951
0.03		0.17818	0.17818	0.21417	0.21417
0.05		0.22520	0.22520	0.28231	0.28231
0.07		0.26222	0.26223	0.33932	0.33931
0.1		0.30756	0.30757	0.41311	0.41308
0.3		0.49640	0.49609	0.76637	0.76698
0.5		0.61628	0.61508	1.02782	1.02984
0.7		0.71282	0.70345	1.24507	1.25826
1.		0.84190	0.79852	1.51109	1.57536
3.		1.64561	0.98358	2.29229	3.58401
5.		2.44561	0.99951	2.45521	5.58408
7.		3.24574	1.00069	2.49226	7.58437
10.		4.44593	1.00030	2.50176	10.58484
30.		12.44718	0.99994	2.49974	30.58796
50.		20.44842	0.99998	2.49983	50.59104
70.		28.44965	0.99999	2.49992	70.59412
100.		40.45150	0.99999	2.49997	100.59861

TABLE 4.5: INTERMEDIATE AND LATE TIME BEHAVIOR FOR LINEAR THICKNESS VARIATION

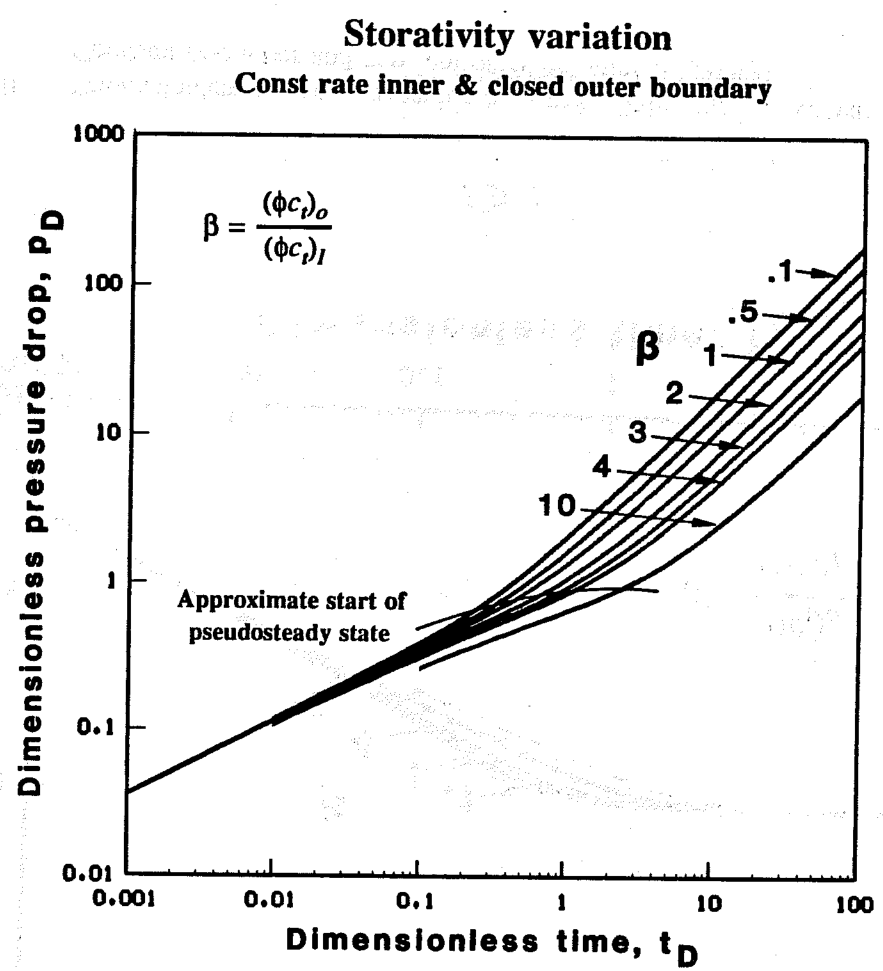
Inner boundary	Outer boundary	Intermediate time behavior	Late time behavior
Constant rate	Closed	$\beta \uparrow, p_D \downarrow$ $\beta \downarrow, p_D \uparrow$	$\beta \uparrow \downarrow, t_{pss} \downarrow$
Constant rate	Constant pressure	$\beta \uparrow, p_D \downarrow$ $\beta \downarrow, p_D \uparrow$	$\beta \uparrow, t_{ss} \downarrow$ $\beta \downarrow, t_{ss} \uparrow$
Constant pressure	Closed	$\beta \uparrow, Q_D \uparrow$ $\beta \downarrow, Q_D \downarrow$	$\beta \uparrow, t_{depletion} \uparrow$ $\beta \downarrow, t_{depletion} \downarrow$
Constant pressure	Constant pressure	$\beta \uparrow, Q_D \uparrow$ $\beta \downarrow, Q_D \downarrow$	$\beta \uparrow, t_{ss} \downarrow$ $\beta \downarrow, t_{ss} \uparrow$

TABLE 4.6: INTERMEDIATE AND LATE TIME BEHAVIOR FOR LINEAR PERMEABILITY VARIATION

Inner boundary	Outer boundary	Intermediate time behavior	Late time behavior
Constant rate	Closed	$\beta \uparrow, p_D \downarrow$ $\beta \downarrow, p_D \uparrow$	Merger to $\beta = 1$ on log-log graph
Constant rate	Constant pressure	$\beta \uparrow, p_D \downarrow$ $\beta \downarrow, p_D \uparrow$	$\beta \uparrow, t_{ss} \downarrow$ $\beta \downarrow, t_{ss} \uparrow$
Constant pressure	Closed	$\beta \uparrow, Q_D \uparrow$ $\beta \downarrow, Q_D \downarrow$	$\beta \uparrow, t_{pss} \downarrow$ $\beta \downarrow, t_{pss} \uparrow$ $Q_D = 1$
Constant pressure	Constant pressure	$\beta \uparrow, Q_D \uparrow$ $\beta \downarrow, Q_D \downarrow$	$\beta \uparrow, t_{ss} \downarrow$ $\beta \downarrow, t_{ss} \uparrow$

TABLE 4.7: INTERMEDIATE AND LATE TIME BEHAVIOR FOR LINEAR POROSITY-COMPRESSIBILITY VARIATION

Inner boundary	Outer boundary	Intermediate time behavior	Late time behavior
Constant rate	Closed	$\beta \uparrow, p_D \downarrow$ $\beta \downarrow, p_D \uparrow$	$\beta \uparrow, t_{pss} \uparrow$ $\beta \downarrow, t_{pss} \downarrow$
Constant rate	Constant pressure	$\beta \uparrow, p_D \downarrow$ $\beta \downarrow, p_D \uparrow$	$\beta \uparrow, t_{ss} \uparrow$ $\beta \downarrow, t_{ss} \downarrow$ $p_D = 1$
Constant pressure	Closed	$\beta \uparrow, Q_D \uparrow$ $\beta \downarrow, Q_D \downarrow$	$\beta \uparrow, t_{pss} \uparrow$ $\beta \downarrow, t_{pss} \downarrow$
Constant pressure	Constant pressure	$\beta \uparrow, Q_D \uparrow$ $\beta \downarrow, Q_D \downarrow$	Merger to $\beta = 1$ on log-log graph



$$p_D = \frac{1}{\beta+1}\left[2t_D - \frac{\beta^2}{\alpha^2}\right]$$

Figure 4.9: Dimensionless pressure behavior for linear, finite aquifer (Linear ϕc_t variation, Constant rate inner and closed outer boundary).

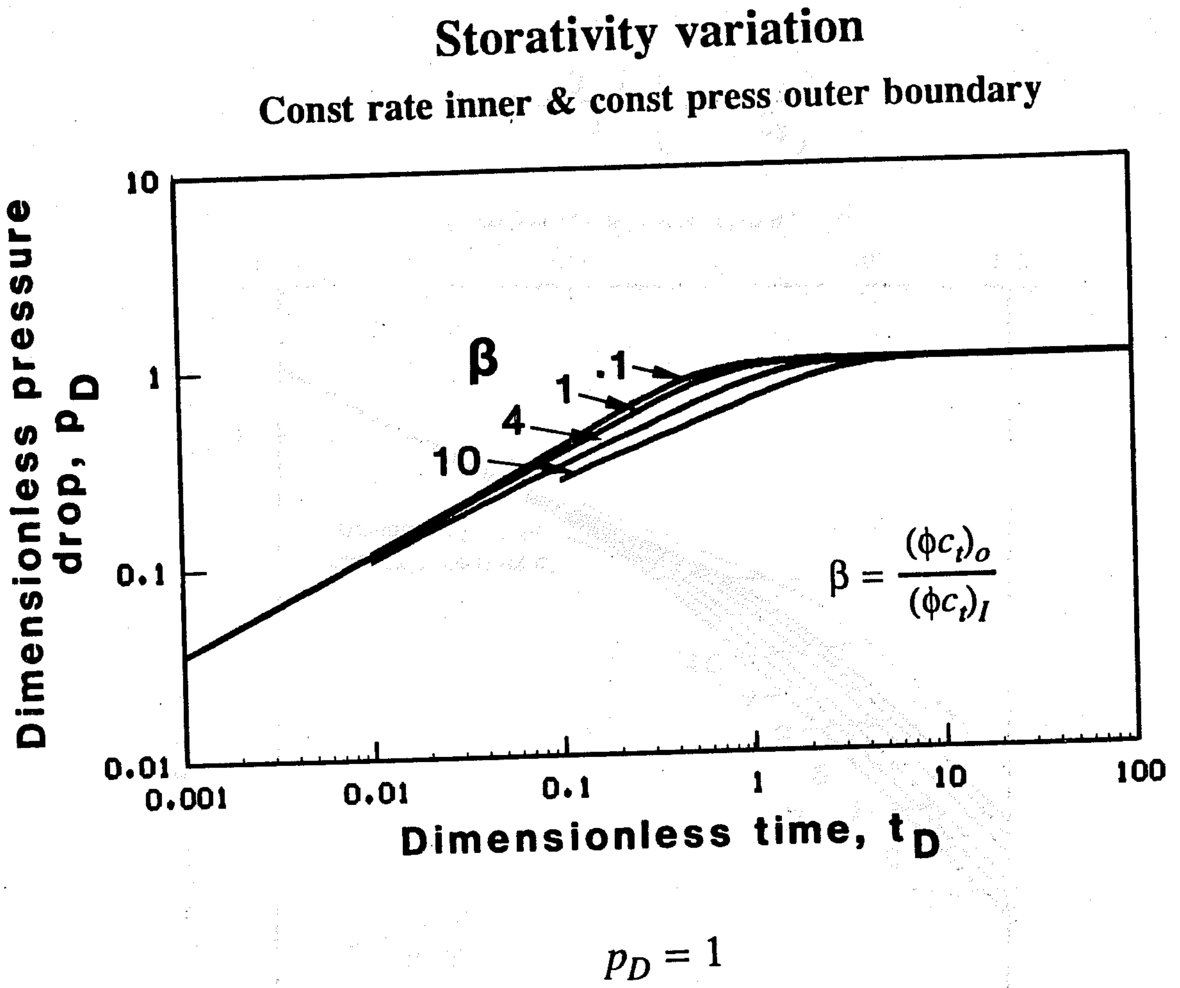
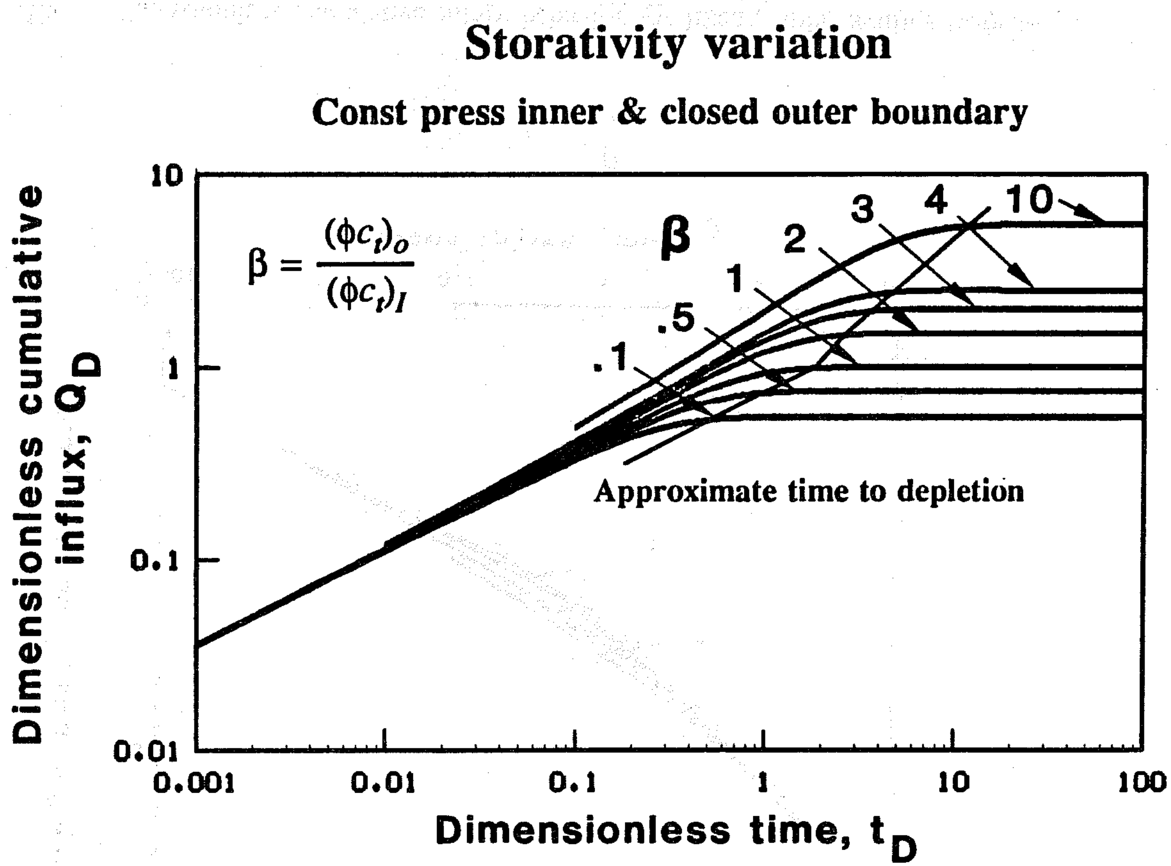


Figure 4.10: Dimensionless pressure behavior for linear, finite aquifer (Linear ϕc_t variation, Constant rate inner and constant pressure outer boundary).

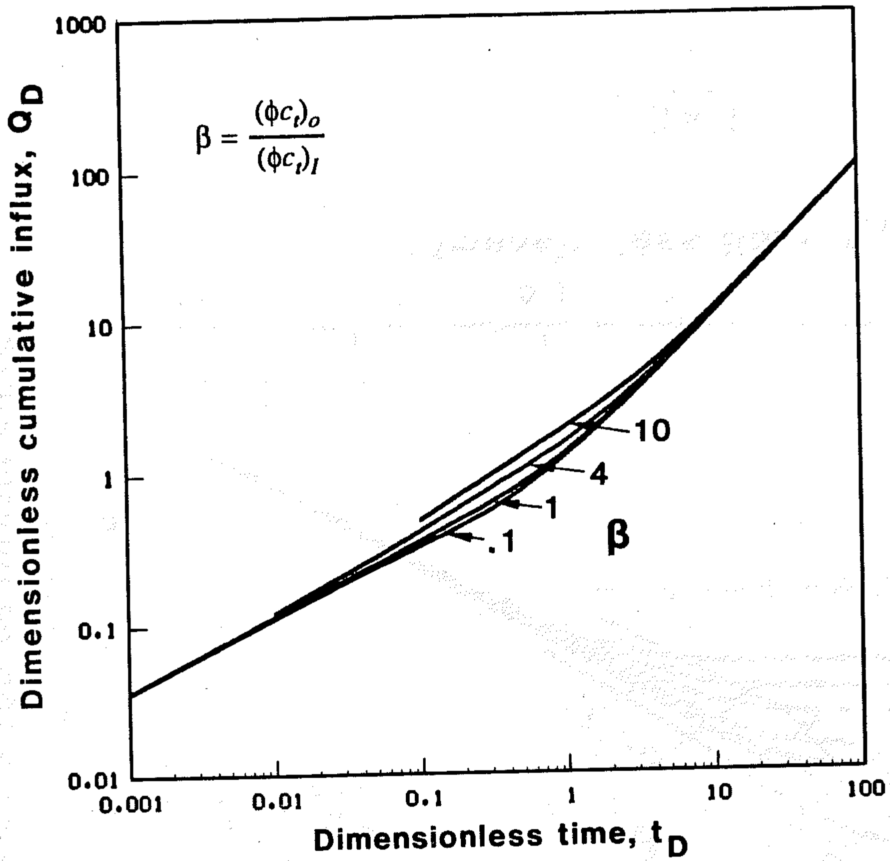


$$Q_D = \frac{\beta + 1}{2}$$

Figure 4.11: Dimensionless cumulative influx behavior for linear, finite aquifer (Linear ϕc_t variation, Constant pressure inner and closed outer boundary).

Storativity variation

Const press inner & outer boundary



$$Q_D = t_D - \frac{\beta}{2\,\alpha^2}$$

Figure 4.12: Dimensionless cumulative influx behavior for linear, finite aquifer (Linear ϕc_t variation, Constant pressure inner and outer boundary).

5. CONCLUSIONS

1. Transient response problems for finite linear aquifers with linearly varying properties have been solved analytically for a variety of boundary conditions.

2. At early times, nonhomogeneous aquifers behave as equivalent homogeneous aquifers. After boundary effects have been felt at the reservoir-aquifer plane, there is a significant transition period before behavior goes to pseudosteady or steady state. During this transition period, response functions differ from the homogeneous aquifer solution. The variation of the response functions depends upon the value of β and type of property variation. Once the aquifer is in pseudosteady or steady state, response functions either merge with or stay parallel to homogeneous aquifer solutions on a log-log graph. Thus nonhomogeneous aquifers do not always necessarily behave as a homogeneous aquifer with equivalent properties.

3. Aquifers attain pseudosteady or steady state at different dimensionless times depending on the type and severity of variation.

4. For constant-rate cases at the inner boundary, the dimensionless pressure drop functions for $\beta < 1$ stay above those for $\beta > 1$; and for constant-pressure cases at the inner boundary, the dimensionless cumulative influx functions for $\beta < 1$ stay below those those for $\beta > 1$.

5. In most cases, if other relevant information is available, a Cartesian graph of pressure drop or cumulative influx vs. time can help determine the value of β except for linear ϕc_t variation (constant-rate inner boundary and constant-pressure outer boundary) and linear permeability variation (constant-pressure inner and closed outer boundary).

6. NOMENCLATURE

A,B,C,D	Constants.
Ai,Bi	Airy functions
a	Parameter defined by Eq. 3.16
a_n	Coefficients defined by Eq. 3.58 for $n \geq 0$
b	Width of the system, and parameter defined by Eq. 3.17
b_n	Coefficients defined by Eq. 3.60 for $n \geq 0$
$C_1 - C_2$	Arbitrary Constants
c_t	Total system compressibility
d,e,g	Parameters in Eq. 3.34
e	Constant in Eq. 3.63
f	Function defined by Eq. 3.15
h	Thickness
h_I	Thickness at inner boundary
h_o	Thickness at outer boundary
I_0,K_0	Modified Bessel function of order zero
I_1,K_1	Modified Bessel function of order one
k	Absolute permeability
k_I	Permeability at inner boundary
L	System length
p	Pressure
p_D	Dimensionless pressure
$p_{D,e}$	Solution to time-independent problem
$p_{D,h}$	Transient contribution to pressure drop
$\overline{p}_D$	Dimensionless pressure drop in Laplace space
p_i	Initial pressure
p_o	Constant flowing pressure
Q_D	Dimensionless cumulative influx
$\overline{Q}_D$	Dimensionless cumulative influx in Laplace space
q	Influx rate
q_D	Dimensionless influx rate
r	Radial distance
r_D	Dimensionless radius
r_e	Outer boundary radius
r_{eD}	Dimensionless outer radius
r_w	Inner boundary radius
r_{wD}	Dimensionless inner boundary radius
s	Laplace variable with respect to time
t	Time
t_D	Dimensionless time
x	Distance
x_D	Dimensionless distance
u	Dependent varible in Eq. 3.34, and parameter in Eq. 3.56
v	Dependent variable in Eq. 3.38
Y_1,Y_2	Independent solutions given by Eqs. 3.57 and 3.59 respectively
z	Parameter as defined by Eq. 2.13, and independent variable in Eq. 3.34

Greek symbols

α	Parameter as defined by Eq. 2.12
α	Function defined by Eq. 3.39 or 3.45
$\hat{\alpha}$	Function of t_D in Eq. 3.67
β	Ratio of property at outer to inner boundary
ε	Parameter as defined by Eq. C-5
γ	Euler's constant
Γ	Parameter as defined by Eq. C-6
∇	Differential operator
∂	Partial derivative
Ψ	Parameter as defined by Eq. C-1
ϕ	Porosity
ϕ	Function of r_D in Eq. 3.67
μ	Viscosity
η	Similarity variable
λ	Separation constant in Eq. 3.68
θ	Variable defined by Eq. 3.42
ξ	Variable defined by Eq. 3.35

Subscripts

D	Dimensionless
i	Initial
I	Inner
o	Outer
t	Total

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APPENDIX A -- LINEAR SYSTEM SOLUTIONS (THICKNESS VARIATION)

The following presents analytical solutions for the cases considered in this study.

Constant-Rate Inner and Closed Outer Boundary

$$\bar{p}_D(x_D, s) = \frac{I_0\left[\frac{\sqrt{s}\,z}{\alpha}\right] K_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] + I_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_0\left[\frac{\sqrt{s}\,z}{\alpha}\right]}{s^{3/2}\left[\pm I_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_1\left[\frac{\sqrt{s}}{\alpha}\right] \mp I_1\left[\frac{\sqrt{s}}{\alpha}\right] K_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right]\right]} \qquad \text{(A-1)}$$

Constant-Rate Inner and Constant-Pressure Outer Boundary

$$\bar{p}_D(x_D, s) = \frac{I_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_0\left[\frac{\sqrt{s}\,z}{\alpha}\right] - I_0\left[\frac{\sqrt{s}\,z}{\alpha}\right] K_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right]}{s^{3/2}\left[\pm I_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_1\left[\frac{\sqrt{s}}{\alpha}\right] \pm I_1\left[\frac{\sqrt{s}}{\alpha}\right] K_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right]\right]} \qquad \text{(A-2)}$$

Constant-Pressure Inner and Closed Outer Boundary

$$\bar{Q}_D(x_D\ s) = \frac{\mp I_1\left[\frac{\sqrt{s}\,z}{\alpha}\right] K_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] \pm I_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_1\left[\frac{\sqrt{s}\,z}{\alpha}\right]}{s^{3/2}\left[I_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_0\left[\frac{\sqrt{s}}{\alpha}\right] + I_0\left[\frac{\sqrt{s}}{\alpha}\right] K_1\left[\frac{\sqrt{s}\,\beta}{\alpha}\right]\right]} \qquad \text{(A-3)}$$

Constant-Pressure Inner and Outer Boundary

$$\bar{Q}_D(x_D, s) = \frac{\pm I_1\left[\frac{\sqrt{s}\,z}{\alpha}\right] K_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] \pm I_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_1\left[\frac{\sqrt{s}\,z}{\alpha}\right]}{s^{3/2}\left[I_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right] K_0\left[\frac{\sqrt{s}}{\alpha}\right] - I_0\left[\frac{\sqrt{s}}{\alpha}\right] K_0\left[\frac{\sqrt{s}\,\beta}{\alpha}\right]\right]} \qquad \text{(A-4)}$$

APPENDIX B -- LINEAR SYSTEM SOLUTIONS (PERMEABILITY VARIATION)

The following presents analytical solutions for the cases considered in this study.

Constant-Rate Inner and Closed Outer Boundary

$$\overline{p}_D(x_D, s) = \frac{I_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_0\left[\frac{2\sqrt{sz}}{\alpha}\right] + I_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]}{s^{3/2}\left[\pm I_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_1\left[\frac{2\sqrt{s}}{\alpha}\right] \mp I_1\left[\frac{2\sqrt{s}}{\alpha}\right]K_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]\right]} \tag{B-1}$$

Constant-Rate Inner and Constant-Pressure Outer Boundary

$$\overline{p}_D(x_D, s) = \frac{I_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_0\left[\frac{2\sqrt{sz}}{\alpha}\right] - I_0\left[\frac{2\sqrt{sz}}{\alpha}\right]K_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]}{s^{3/2}\left[\pm I_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_1\left[\frac{2\sqrt{s}}{\alpha}\right] \pm I_1\left[\frac{2\sqrt{s}}{\alpha}\right]K_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]\right]} \tag{B-2}$$

Constant-Pressure Inner and Closed Outer Boundary

$$\overline{Q}_D(x_D, s) = \frac{\pm I_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_1\left[\frac{2\sqrt{sz}}{\alpha}\right] \mp I_1\left[\frac{2\sqrt{sz}}{\alpha}\right]K_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]}{\sqrt{z}\, s^{3/2}\left[I_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_0\left[\frac{2\sqrt{s}}{\alpha}\right] + I_0\left[\frac{2\sqrt{s}}{\alpha}\right]K_1\left[\frac{2\sqrt{s\beta}}{\alpha}\right]\right]} \tag{B-3}$$

Constant-Pressure Inner and Outer Boundary

$$\overline{Q}_D(x_D, s) = \frac{\pm I_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_1\left[\frac{2\sqrt{sz}}{\alpha}\right] \pm I_1\left[\frac{2\sqrt{sz}}{\alpha}\right]K_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]}{\sqrt{z}\, s^{3/2}\left[I_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]K_0\left[\frac{2\sqrt{s}}{\alpha}\right] - I_0\left[\frac{2\sqrt{s}}{\alpha}\right]K_0\left[\frac{2\sqrt{s\beta}}{\alpha}\right]\right]} \tag{B-4}$$

These solutions also apply in cases of linearly varying permeability-viscosity ratio or reciprocal viscosity.

APPENDIX C -- LINEAR SYSTEM SOLUTIONS (ϕc_t VARIATION)

Equation 2.33 can be transformed into standard Airy equation form by substituting:

$$\Psi = \left[\frac{\sqrt{s}}{\alpha}\right]^{2/3} z \,, \tag{C-1}$$

which then becomes:

$$\frac{d^2\overline{p}_D}{d\Psi^2} - \Psi\overline{p}_D = 0 \,, \tag{C-2}$$

whose solution is:

$$\overline{p}_D = A \;\; Ai(\Psi) + B \;\; Bi(\Psi) \,, \tag{C-3}$$

where A and B are constants determined by boundary conditions. Airy functions can be represented in terms of fractional Bessel functions (Abramowitz and Stegun, 1972). Solutions for different cases follow:

Constant-Rate Inner and Closed Outer Boundary

$$\overline{p}_D(x_D, s) = \frac{Ai'(\varepsilon)Bi(\Psi) - Ai(\Psi)Bi'(\varepsilon)}{s(s\alpha)^{1/3}\left[\mp Ai'(\varepsilon)Bi'(\Gamma) \pm Ai'(\Gamma)Bi'(\varepsilon)\right]} \tag{C-4}$$

where:

$$\varepsilon = \left[\frac{\sqrt{s}}{\alpha}\right]^{2/3} \beta \tag{C-5}$$

$$\Gamma = \left[\frac{\sqrt{s}}{\alpha}\right]^{2/3} \tag{C-6}$$

and Ai' and Bi' refer to derivatives of Airy functions with respect to z.

Constant-Rate Inner and Constant-Pressure Outer Boundary

$$\overline{p}_D(x_D, s) = \frac{Ai(\varepsilon)Bi(\Psi) - Ai(\Psi)Bi(\varepsilon)}{s(s\alpha)^{1/3}\left[\mp Ai(\varepsilon)Bi'(\Gamma) \pm Ai'(\Gamma)Bi(\varepsilon)\right]} \tag{C-7}$$

Constant-Pressure Inner and Closed Outer Boundary

$$\overline{Q}_D(x_D,s) = \frac{(s\alpha)^{1/3}}{s^2}\frac{\left[\mp Ai'(\varepsilon)Bi'(\Psi) \pm Ai'(\Psi)Bi'(\varepsilon)\right]}{\left[Ai'(\varepsilon)Bi(\Gamma) - Bi'(\varepsilon)Ai(\Gamma)\right]} \qquad \text{(C-8)}$$

Constant-Pressure Inner and Outer Boundary

$$\overline{Q}_D(x_D,\, s) = \frac{(s\alpha)^{1/3}}{s^2}\frac{\left[\mp Ai(\varepsilon)Bi'(\Psi) \pm Ai'(\Psi)Bi(\varepsilon)\right]}{\left[Ai(\varepsilon)Bi(\Gamma) - Bi(\varepsilon)Ai(\Gamma)\right]} \qquad \text{(C-9)}$$

APPENDIX D -- LINEAR HOMOGENEOUS SYSTEM SOLUTIONS

In the case of a linear homogeneous system ($\beta = 1$), the partial differential equation is:

$$\frac{\partial^2 p_D}{\partial x_D^2} = \frac{\partial p_D}{\partial t_D} \tag{D-1}$$

Solutions for different cases can be obtained via Laplace transform and are presented in the following.

Constant-Rate Inner and Closed Outer Boundary

$$\bar{p}_D(x_D, s) = \frac{e^{\sqrt{s}(1 - x_D)} + e^{-\sqrt{s}(1 - x_D)}}{s^{3/2}\,(e^{\sqrt{s}} - e^{-\sqrt{s}})} \tag{D-2}$$

Constant-Rate Inner and Constant-Pressure Outer Boundary

$$\bar{p}_D(x_D, s) = \frac{e^{\sqrt{s}(1 - x_D)} - e^{-\sqrt{s}(1 - x_D)}}{s^{3/2}(e^{\sqrt{s}} + e^{-\sqrt{s}})} \tag{D-3}$$

Constant-Pressure Inner and Closed Outer Boundary

$$\bar{Q}_D(x_D, s) = \frac{e^{\sqrt{s}(1 - x_D)} - e^{-\sqrt{s}(1 - x_D)}}{s^{3/2}(e^{\sqrt{s}} + e^{-\sqrt{s}})} \tag{D-4}$$

Constant-Pressure Inner and Outer Boundary

$$\bar{Q}_D(x_D, s) = \frac{e^{\sqrt{s}(1 - x_D)} + e^{-\sqrt{s}(1 - x_D)}}{s^{3/2}\,(e^{\sqrt{s}} - e^{-\sqrt{s}})} \tag{D-5}$$

The preceding solutions were evaluated at $x_D = 0$, corresponding to the reservoir-aquifer plane. Equations D-2 and D-5 will produce the same solution. The same remark is true for Eqs. D-3 and D-4.

APPENDIX E -- EARLY AND LATE TIME APPROXIMATE SOLUTIONS FOR LINEAR SYSTEMS

The following presents early and late time solutions for the cases considered in this study.

Early Time Solutions

Early time solutions were obtained by using the approximations for Bessel and Airy functions as $s \rightarrow \infty$ (Abramowitz and Stegun, 1972). Solutions were then inverted analytically. Early time solutions for all cases are:

$$p_D \text{ or } (Q_D) = 2\sqrt{\frac{t_D}{\pi}} \tag{E-1}$$

Late Time Solutions

Late time solutions in Laplace space were derived using the approximations for Bessel and Airy functions as $s \rightarrow 0$(Abramowitz and Stegun, 1972). These solutions were then inverted analytically. Late time solutions are listed in the following.

Linear Thickness Variation

Late time solutions for different boundary conditions for the case of linear thickness variation follow:

Constant-Rate Inner and Closed Outer Boundary

$$p_D = \frac{1}{(\beta+1)}\left[2t_D - \left[\frac{\beta}{\beta-1}\right]^2 [0.5 - ln(\alpha)]\right] \tag{E-2}$$

Constant-Rate Inner and Constant-Pressure Outer Boundary

$$p_D = \frac{\ln(\beta)}{\beta-1} \tag{E-3}$$

Constant-Pressure Inner and Closed Outer Boundary

$$Q_D = \frac{\beta+1}{2} \tag{E-4}$$

Constant-Pressure Inner and Outer Boundary

$$Q_D = \pm \frac{\alpha}{\ln(\beta)} t_D \pm \frac{\left[-1+2\ln(2\alpha/\beta)-2\gamma\right]}{4\alpha\, ln(\beta)} \tag{E-5}$$

where γ is Euler's constant, equal to 0.577216.

Linear Permeability Variation

When permeability or mobility varies linearly with distance, late time solutions are given by the following:

Constant-Rate Inner and Closed Outer Boundary

$$p_D = t_D + \frac{\beta}{\alpha^2}\left[1+2ln(\alpha)-2\gamma\right] \tag{E-6}$$

Constant-Rate Inner and Constant-Pressure Outer Boundary

$$p_D = \frac{ln(\beta)}{\beta-1} \tag{E-7}$$

Constant-Pressure Inner and Closed Outer Boundary

$$Q_D = 1 \tag{E-8}$$

Constant-Pressure Inner and Outer Boundary

$$Q_D = \pm\frac{\alpha}{ln(\beta)} t_D \pm \frac{\left[1-2\ln(\sqrt{\beta}/\alpha)-2\gamma\right]}{\alpha\, ln(\beta)} \tag{E-9}$$

Linear Porosity-compressibility Product Variation

Late time solutions for the case of ϕc_t variation are presented in the following;

Constant-Rate Inner and Closed Outer Boundary

$$p_D = \frac{1}{\beta+1}\left[2t_D - \frac{\beta^2}{\alpha^2}\right] \tag{E-10}$$

Constant-Rate Inner and Constant-Pressure Outer Boundary

$$p_D = 1 \tag{E-11}$$

Constant-Pressure Inner and Closed Outer Boundary

$$Q_D = \frac{\beta + 1}{2} \tag{E-12}$$

Constant-Pressure Inner and Outer Boundary

$$Q_D = t_D - \frac{\beta}{2\,\alpha^2} \tag{E-13}$$

APPENDIX F

Program No. 1 --- Linear system with linear thickness variation
Program No. 2 --- Linear system with linear permeability variation
Program No. 3 --- Linear system with linear porosity-compressibility variation
Program No. 4 --- Radial system with permeability variation linear with r
Program No. 5 --- Stehfest algorithm used with Program No. 1, 2, and 3
Program No. 6 --- Stehfest algorithm used with Program No. 4

```
*         ************************************************
*         *               PROGRAM # 1                   *
*         *               -------------                 *
*         *   Name : Anil Kumar Ambastha                *
*         *   Date : May      30, 1985                  *
*         *                                             *
*         *   Purpose of this program is to generate    *
*         *   pressure transient response at the inner  *
*         *   boundary in non-homogeneous linear system.*
*         *       This program handles cases for beta   *
*         *       greater and less than 1 and deals     *
*         *       with linear variation of thickness    *
*         *       in a linear system. It can handle     *
*         *       IBC of const rate or const pre. and   *
*         *       OBC of closed or const. pre.          *
*         ************************************************
          IMPLICIT REAL*8 (A-H,O-Z)
          DIMENSION TD(20)
          COMMON BETA,M,ICODE,JCODE
          OPEN(UNIT=7,FILE="output")
          PRINT *,'READ THE VALUE OF BETA '
          READ(5,*)BETA
          PRINT *,'# OF CYCLES OF DATA REQUIRED'
          READ(5,*)NC
          PRINT *,'GIVE FIRST VALUE OF TD'
          READ(5,*)TD1
          PRINT *,'NUMBER OF TERMS TO BE USED IN STEHFEST'
          READ(5,*)NTERM

C         READ CODES FOR BOUNDARY CONDITIONS

          PRINT *,'SUPPLY INNER BOUNDARY CONDITION CODE'
          PRINT *,'1 ---- CONSTANT RATE'
          PRINT *,'2 ---- CONSTANT PRESSURE'
          READ(5,*)ICODE

          PRINT *,'SUPPLY OUTER BOUNDARY CONDITION CODE'
          PRINT *,'1 ---- CLOSED'
          PRINT *,'2 ---- CONSTANT PRESSURE'
          READ(5,*)JCODE
          M=777

C         GENERATE THE FIRST SET OF TD VECTOR

          TD(1)=TD1
          TD(2)=1.5*TD1
          TD(3)=2.*TD1
          TD(4)=2.5*TD1
          TD(5)=3.*TD1
          TD(6)=3.5*TD1
          TD(7)=4.*TD1
          TD(8)=4.5*TD1
          TD(9)=5.*TD1
          TD(10)=6.*TD1
          TD(11)=7.*TD1
          TD(12)=8.*TD1
          TD(13)=9.*TD1

C         WRITE THE NUMBER OF DATA POINTS GENERATED
```

```
      WRITE(7,*)13*NC
C     GENERATE AND PRINT THE PRESSURE TRANSIENT RESPONSE
      DO 1 I=1,NC
      DO 2 J=1,13
      SPC=TD(J)
      PD=PWD(SPC,NTERM)
      WRITE(7,9)TD(J),PD
  2   TD(J)=10.*TD(J)
  1   CONTINUE
  9   FORMAT(2X,F15.5,2X,F15.7)
      STOP
      END
*********************************************************************
      FUNCTION PWDL(S)
      IMPLICIT REAL*8 (A-H,O-Z)
      DOUBLE PRECISION MMBSI0,MMBSI1,MMBSK0,MMBSK1
      COMMON BETA,M,ICODE,JCODE

      IF(BETA.GT.1.)ALPHA=BETA-1.
      IF(BETA.LT.1.)ALPHA=1.-BETA
      ARG1=BETA*DSQRT(S)/ALPHA
      ARG2=DSQRT(S)/ALPHA


      X1=DSQRT(S)
      X2=S**1.5

C     EVALUATION OF CONSTANTS 'A' AND 'B'

      F=DEXP(-2.*(ARG1-ARG2))

      IF(JCODE.EQ.1)THEN
         C1=X1*MMBSI1(2,ARG1,IER)
         C2=-X1*MMBSK1(2,ARG1,IER)
      ENDIF

      IF(JCODE.EQ.2)THEN
         C1=MMBSI0(2,ARG1,IER)
         C2=MMBSK0(2,ARG1,IER)
      ENDIF

C     CALCULATION OF C3 AND C4 FOR ICODE = 1

*     APPLIES TO BETA > 1.

      IF(ICODE.EQ.1)THEN
         C3=-X2*MMBSI1(2,ARG2,IER)
         C4=X2*MMBSK1(2,ARG2,IER)
      ENDIF

*     APPLIES TO BETA < 1.

      IF(ICODE.EQ.1.AND.BETA.LT.1.)THEN
         C3=-C3
         C4=-C4
      ENDIF

C     CALCULAION OF C3 AND C4 FOR ICODE = 2

      IF(ICODE.EQ.2)THEN
         C3=S*MMBSI0(2,ARG2,IER)
```

```
         C4=S*MMBSK0(2,ARG2,IER)
      ENDIF


C     CALCULATION OF A1 AND A2 FOR ICODE = 1

      IF(ICODE.EQ.1)THEN
      A1=MMBSI0(2,ARG2,IER)
      A2=MMBSK0(2,ARG2,IER)
      ENDIF

C     CALCULATION OF A1 AND A2 FOR ICODE = 2

      IF(ICODE.EQ.2)THEN
         A1=-MMBSI1(2,ARG2,IER)/X1
         A2=MMBSK1(2,ARG2,IER)/X1
      ENDIF

      IF(ICODE.EQ.2.AND.BETA.LT.1.)THEN
         A1=-A1
         A2=-A2
      ENDIF


C     CALCULATION OF THE TRANSFORMED SOLUTION

*     FOR BETA > 1., IT IS AT XD = 0.
*     FOR BETA < 1., IT IS AT XD = 1.


      F1=C4-C2*C3*F/C1
      F2=-C2*A1*F/C1/F1
      F3=A2/F1

C     PWDL REPRESENTS LAPLACE TRANSFORM OF PWD FOR ICODE = 1
C     PWDL REPRESENTS LAPLACE TRANSFORM OF QD FOR ICODE = 2


      PWDL=F2+F3
      RETURN
      END
*****************************************************************************
```

```
*       ************************************************
*       *                PROGRAM No. 2                 *
*       *               -----------------              *
*       *   Name : Anil Kumar Ambastha                 *
*       *   Date : May      30, 1985                   *
*       *                                              *
*       *   Purpose of this program is to generate     *
*       *   pressure transient response at the inner   *
*       *   boundary in non-homogeneous linear system. *
*       *       This program handles cases for beta    *
*       *       greater and less than 1 and deals      *
*       *       with linear variation of permeability  *
*       *       in a linear system. It can handle      *
*       *       IBC of const rate or const pre. and    *
*       *       OBÇ of closed or const. pre.           *
*       ************************************************
        IMPLICIT REAL*8 (A-H,O-Z)
        DIMENSION TD(20)
        COMMON BETA,M,ICODE,JCODE
        OPEN(UNIT=7,FILE="output")
        PRINT *,'READ THE VALUE OF BETA '
        READ(5,*)BETA
        PRINT *,'# OF CYCLES OF DATA REQUIRED'
        READ(5,*)NC
        PRINT *,'GIVE FIRST VALUE OF TD'
        READ(5,*)TD1
        PRINT *,'NUMBER OF TERMS TO BE USED IN STEHFEST'
        READ(5,*)NTERM

C       READ CODES FOR BOUNDARY CONDITIONS

        PRINT *,'SUPPLY INNER BOUNDARY CONDITION CODE'
        PRINT *,'1 ---- CONSTANT RATE'
        PRINT *,'2 ---- CONSTANT PRESSURE'
        READ(5,*)ICODE

        PRINT *,'SUPPLY OUTER BOUNDARY CONDITION CODE'
        PRINT *,'1 ---- CLOSED'
        PRINT *,'2 ---- CONSTANT PRESSURE'
        READ(5,*)JCODE
        M=777

C       GENERATE THE FIRST SET OF TD VECTOR

        TD(1)=TD1
        TD(2)=1.5*TD1
        TD(3)=2.*TD1
        TD(4)=2.5*TD1
        TD(5)=3.*TD1
        TD(6)=3.5*TD1
        TD(7)=4.*TD1
        TD(8)=4.5*TD1
        TD(9)=5.*TD1
        TD(10)=6.*TD1
        TD(11)=7.*TD1
        TD(12)=8.*TD1
        TD(13)=9.*TD1
```

```
C       WRITE THE NUMBER OF DATA POINTS GENERATED

        WRITE(7,*)13*NC

C       GENERATE AND PRINT THE PRESSURE TRANSIENT RESPONSE

        DO 1 I=1,NC
        DO 2 J=1,13
        SPC=TD(J)
        PD=PWD(SPC,NTERM)
        WRITE(7,9)TD(J),PD
 2      TD(J)=10.*TD(J)
 1      CONTINUE
 9      FORMAT(2X,F15.5,2X,F15.7)
        STOP
        END
******************************************************************

        FUNCTION PWDL(S)
        IMPLICIT REAL*8 (A-H,O-Z)
        DOUBLE PRECISION MMBSI0,MMBSI1,MMBSK0,MMBSK1
        COMMON BETA,M,ICODE,JCODE

        IF(BETA.GT.1.)ALPHA=BETA-1.
        IF(BETA.LT.1.)ALPHA=1.-BETA
        ARG1=2./ALPHA*DSQRT(S*BETA)
        ARG2=2./ALPHA*DSQRT(S)


        X1=DSQRT(S/BETA)
        X2=S**1.5

C       EVALUATION OF CONSTANTS 'A' AND 'B'

        F=DEXP(-2.*(ARG1-ARG2))
        IF(JCODE.EQ.1)THEN
            C1=X1*MMBSI1(2,ARG1,IER)
            C2=-X1*MMBSK1(2,ARG1,IER)
        ENDIF

        IF(JCODE.EQ.2)THEN
            C1=MMBSI0(2,ARG1,IER)
            C2=MMBSK0(2,ARG1,IER)
        ENDIF

C       CALCULATION OF C3 AND C4 FOR ICODE = 1

*       APPLIES TO BETA > 1

        IF(ICODE.EQ.1)THEN
            C3=-X2*MMBSI1(2,ARG2,IER)
            C4=X2*MMBSK1(2,ARG2,IER)
        ENDIF

*       APPLIES TO BETA < 1

        IF(ICODE.EQ.1.AND.BETA.LT.1.)THEN
           C3=-C3
           C4=-C4
        ENDIF

C       CALCULATION OF C3 AND C4 FOR ICODE = 2

        IF(ICODE.EQ.2)THEN
           C3=S*MMBSI0(2,ARG2,IER)
```

```
         C4=S*MMBSK0(2,ARG2,IER)
      ENDIF

C     CALCULATION OF A1 AND A2 FOR ICODE = 1

      IF(ICODE.EQ.1)THEN
         A1=MMBSI0(2,ARG2,IER)
         A2=MMBSK0(2,ARG2,IER)
      ENDIF

C     CALCULATION OF A1 AND A2 FOR ICODE = 2

      IF(ICODE.EQ.2)THEN
         A1=-MMBSI1(2,ARG2,IER)/DSQRT(S)
         A2=MMBSK1(2,ARG2,IER)/DSQRT(S)
      ENDIF

      IF(ICODE.EQ.2.AND.BETA.LT.1.)THEN
         A1=-A1
         A2=-A2
      ENDIF

C     CALCULATION OF TRANSFORMED SOLUTION

*     FOR BETA > 1., IT IS AT XD =  0
*     FOR BETA < 1., IT IS AT XD = 1.

      F1=C4-C2*C3*F/C1
      F2=-C2*A1*F/C1/F1
      F3=A2/F1

C     PWDL REPRESENTS LAPLACE TRANSFORM OF PWD FOR ICODE = 1
C     PWDL REPRESNTS LAPLACE TRANSFORM OF QD FOR ICODE = 2

      PWDL=F2+F3
      RETURN
      END
***********************************************************************
```

```
*       **************************************************
*       *                 PROGRAM No. 3                  *
*       *                 ---------------                *
*       *   Name : Anil Kumar Ambastha                   *
*       *   Date : June     18, 1985                     *
*       *                                                *
*       *   Purpose of this program is to generate       *
*       *   pressure transient response at the inner    *
*       *   boundary in non-homogeneous linear system.   *
*       *       This program handles cases for beta      *
*       *       greater and less than 1 and deals        *
*       *       with linear variation of porosity * comp. *
*       *       in a linear system. It can handle        *
*       *       IBC of const rate or const pre. and      *
*       *       OBC of closed or const. pre.             *
*       **************************************************
        IMPLICIT REAL*8 (A-H,O-Z)
        DIMENSION TD(20)
        COMMON BETA,M,ICODE,JCODE
        OPEN(UNIT=7,FILE="output")
        PRINT *,'READ THE VALUE OF BETA '
        READ(5,*)BETA
        PRINT *,'# OF CYCLES OF DATA REQUIRED'
        READ(5,*)NC
        PRINT *,'GIVE FIRST VALUE OF TD'
        READ(5,*)TD1
        PRINT *,'NUMBER OF TERMS TO BE USED IN STEHFEST'
        READ(5,*)NTERM

C       READ CODES FOR BOUNDARY CONDITIONS

        PRINT *,'SUPPLY INNER BOUNDARY CONDITION CODE'
        PRINT *,'1 ---- CONSTANT RATE'
        PRINT *,'2 ---- CONSTANT PRESSURE'
        READ(5,*)ICODE

        PRINT *,'SUPPLY OUTER BOUNDARY CONDITION CODE'
        PRINT *,'1 ---- CLOSED'
        PRINT *,'2 ---- CONSTANT PRESSURE'
        READ(5,*)JCODE
        M=777

C       GENERATE THE FIRST SET OF TD VECTOR

        TD(1)=TD1
        TD(2)=1.5*TD1
        TD(3)=2.*TD1
        TD(4)=2.5*TD1
        TD(5)=3.*TD1
        TD(6)=3.5*TD1
        TD(7)=4.*TD1
        TD(8)=4.5*TD1
        TD(9)=5.*TD1
        TD(10)=6.*TD1
        TD(11)=7.*TD1
        TD(12)=8.*TD1
        TD(13)=9.*TD1
```

```
C         WRITE THE NUMBER OF DATA POINTS GENERATED

          WRITE(7,*)13*NC

C         GENERATE AND PRINT THE PRESSURE TRANSIENT RESPONSE

          DO 1 I=1,NC
          DO 2 J=1,13
          SPC=TD(J)
          PD=PWD(SPC,NTERM)
          WRITE(7,9)TD(J),PD
   2      TD(J)=10.*TD(J)
   1      CONTINUE
   9      FORMAT(2X,F15.5,2X,F15.7)
          STOP
          END
*******************************************************************

          FUNCTION PWDL(S)
          IMPLICIT REAL*8 (A-H,O-Z)
          DIMENSION AK(1),AI(1)
          COMMON BETA,M,ICODE,JCODE

          IF(BETA.GT.1.)ALPHA=BETA-1.
          IF(BETA.LT.1.)ALPHA=1.-BETA
          C1=1./3.
          C=2./3.
          X=DSQRT(S)/ALPHA
          PI=2.*ASIN(1.)
          ARG2=C*X
          ARG1=ARG2*BETA**1.5
          Z2=X**C
          Z1=BETA*Z2

          X1=ALPHA*X**C
          X2=X1*S
          X3=X1/S
C         X4=DEXP(-2.*ARG1)
C         X5=DEXP(-2.*ARG2)

          CALL MMBSKR(ARG2,C1,1,AK,IER)
          CALL MMBSIR(ARG2,C1,1,1,AI,IER)
          ak(1)=ak(1)*dexp(-arg2)
          AI2=DSQRT(Z2/3.)/PI*AK(1)
          BI2=DSQRT(Z2/3.)*(DSQRT(3.D00)/PI*AK(1)+2.*
     &    AI(1))


          CALL MMBSKR(ARG2,C,1,AK,IER)
          CALL MMBSIR(ARG2,C,1,1,AI,IER)
          ak(1)=ak(1)*dexp(-arg2)
          AIP2=-Z2/PI/DSQRT(3.D00)*AK(1)
          BIP2=Z2/DSQRT(3.D00)*(2.*AI(1)+DSQRT(3.D00)/
     &    PI*AK(1))

C         EVALUATION OF CONSTANTS 'A' AND 'B'

C         F=DEXP(-2.*(ARG1-ARG2))

          IF(JCODE.EQ.1)THEN
             CALL MMBSKR(ARG1,C,1,AK,IER)
             CALL MMBSIR(ARG1,C,1,1,AI,IER)
             ak(1)=ak(1)*dexp(-arg1)
             AIP1=-Z1/PI/DSQRT(3.D00)*AK(1)
```

```
         BIP1=Z1/DSQRT(3.D00)*(2.*AI(1)+DSQRT(3.D00)/
     &   PI*AK(1))
         C1=X1*AIP1
         C2=X1*BIP1
      ENDIF

      IF(JCODE.EQ.2)THEN
         CALL MMBSKR(ARG1,C1,1,AK,IER)
         CALL MMBSIR(ARG1,C1,1,1,AI,IER)
         ak(1)=ak(1)*dexp(-arg1)
         AI1=DSQRT(Z1/3.)/PI*AK(1)
         BI1=DSQRT(Z1/3.)*(DSQRT(3.D00)/PI*AK(1)+2.*AI(1))
         C1=AI1
         C2=BI1
      ENDIF

C     CALCULATION OF C3 AND C4 FOR ICODE = 1

*     APPLIES TO BETA > 1.

      IF(ICODE.EQ.1)THEN
          C3=-X2*AIP2
          C4=-X2*BIP2
      ENDIF

*     APPLIES TO BETA < 1.

      IF(ICODE.EQ.1.AND.BETA.LT.1.)THEN
         C3=-C3
         C4=-C4
      ENDIF

C     CALCULAION OF C3 AND C4 FOR ICODE = 2

      IF(ICODE.EQ.2)THEN
         C3=AI2*S
         C4=BI2*S
      ENDIF


C     CALCULATION OF A1 AND A2 FOR ICODE = 1

      IF(ICODE.EQ.1)THEN
      A1=AI2
      A2=BI2
      ENDIF

C     CALCULATION OF A1 AND A2 FOR ICODE = 2

      IF(ICODE.EQ.2)THEN
         A1=-X3*AIP2
         A2=-X3*BIP2
      ENDIF

      IF(ICODE.EQ.2.AND.BETA.LT.1.)THEN
         A1=-A1
         A2=-A2
      ENDIF


C     CALCULATION OF THE TRANSFORMED SOLUTION

*     FOR BETA > 1., IT IS AT XD = 0.
*     FOR BETA < 1., IT IS AT XD = 1.
```

```
      F1=C4-C2*C3/C1
      F2=-C2*A1/C1/F1
      F3=A2/F1

C     PWDL REPRESENTS LAPLACE TRANSFORM OF PWD FOR ICODE = 1
C     PWDL REPRESENTS LAPLACE TRANSFORM OF QD FOR ICODE = 2


      PWDL=F2+F3
      RETURN
      END
**********************************************************************
```

```
*       ************************************************
*       *                PROGRAM # 4                   *
*       *                -------------                 *
*       *   Name : Anil Kumar Ambastha                 *
*       *   Date : March  26, 1985                     *
*       *                                              *
*       *   Purpose of this program is to generate     *
*       *   pressure transient response at the well    *
*       *   in non-homogeneous radial system.          *
*       *   BETA is the ratio of permeability at the   *
*       *   outer boundary to that at the inner boundary. *
*       *                                              *
*       *   Inner boundary:  Constant rate             *
*       *   Outer boundary:  Closed                    *
*       ************************************************
        IMPLICIT REAL*8 (A-H,O-Z)
        DIMENSION TD(20),TDC(20)
        COMMON RWD,BETA,TOLR,M
        OPEN(UNIT=7,FILE="OUTPUT")
        OPEN(UNIT=8,FILE="CO1")
        OPEN(UNIT=9,FILE="CO2")
        PRINT *,'READ THE VALUES OF RWD,BETA AND TOLERANCE'
        READ(5,*)RWD,BETA,TOLR
        PRINT *,'# OF CYCLES OF DATA REQUIRED'
        READ(5,*)NC
        PRINT *,'GIVE FIRST VALUE OF TD'
        READ(5,*)TD1
        PRINT *,'NUMBER OF TERMS TO BE USED IN STEHFEST'
        READ(5,*)NTERM
        M=777

C       GENERATE THE FIRST SET OF TD VECTOR

        RR=RWD**2
        TD(1)=TD1*RR
        TD(2)=1.5*TD1*RR
        TD(3)=2.*TD1*RR
        TD(4)=2.5*TD1*RR
        TD(5)=3.*TD1*RR
        TD(6)=3.5*TD1*RR
        TD(7)=4.*TD1*RR
        TD(8)=4.5*TD1*RR
        TD(9)=5.*TD1*RR
        TD(10)=6.*TD1*RR
        TD(11)=7.*TD1*RR
        TD(12)=8.*TD1*RR
        TD(13)=9.*TD1*RR

C       WRITE THE NUMBER OF DATA POINTS GENERATED

        WRITE(7,*)13*NC

C       GENERATE AND PRINT THE PRESSURE TRANSIENT RESPONSE

        DO 1 I=1,NC
        DO 2 J=1,13
        SPC=TD(J)
        TDC(J)=TD(J)/RR
        PD=PWD(SPC,NTERM)
```

```
      WRITE(7,*)TDC(J),PD
  2   TD(J)=10.*TD(J)
  1   CONTINUE
      STOP
      END
*******************************************************************

      FUNCTION PWDL(S)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION AB(0:100),B(0:100)
      COMMON RWD,BETA,TOLR,M


C     CALCULATION OF ARRAY 'AB'

      S1=1.-RWD
      S2=1.-BETA*RWD
      AB(0)=1.
      AB(1)=0.
      IF(BETA.NE.1.)THEN
         DO 13 N=2,100
       AB(N)=(S*S1*AB(N-2)-(BETA-1.)*N*(N-1)*AB(N-1))/N/N/S2
      write(8,*)n,ab(n)
 13   CONTINUE
      ELSE
         DO 15 N=2,100,2
         AB(N)=S*AB(N-2)/N/N
 15   CONTINUE
         DO 16 N=3,99,2
 16      AB(N)=0.
      ENDIF

C     CALCULATION OF ARRAY 'B'

      B(0)=0.
      B(1)=(1.-BETA)/S2
      IF(BETA.NE.1.)THEN
      DO 14 N=2,100
      FAC1=(N*B(N-2)-2.*AB(N-2))/(N**3)*S1/S2
      FAC2=(N*(AB(N-1)+(N-1)*B(N-1))-(N-1)*AB(N-1))*B(1)/N/N
      B(N)=S*FAC1+FAC2
      write(9,*)n,b(n)
 14   CONTINUE
      ELSE
         DO 17 N=2,100,2
         B(N)=S*((N*B(N-2)-2.*AB(N-2))/N**3)
 17   CONTINUE
         DO 18 N=3,99,2
 18      B(N)=0.
      ENDIF


C     CALCULATION OF 'Y1'
      N=2
      IF(BETA.NE.1.)MM=-999
      IF(BETA.NE.1.)SUM=1.
      IF(BETA.EQ.1.)SUM=1.+AB(2)*RWD**2
  1   SUM1=SUM
      IF(BETA.NE.1.)SUM=SUM+AB(N)*RWD**N
      IF(BETA.EQ.1.)THEN
         MM=2*N
         SUM=SUM+AB(MM)*RWD**MM
      ENDIF
      IF(DABS((SUM-SUM1)/SUM).LE.TOLR)THEN
        Y1=SUM
```

```
            GO TO 2
          ELSE
            IF(N.GE.60.OR.MM.GE.60)THEN
               Y1=SUM
               PRINT *,'ITR LIM EXCEEDED FOR Y1'
               GO TO 2
          ELSE
            N=N+1
            GO TO 1
          ENDIF
          ENDIF

C         CALCULATION OF 'Y2'

  2       N=2
          XX=DLOG(RWD)
          IF(BETA.NE.1.)SUM=XX+B(1)*RWD
          IF(BETA.EQ.1.)SUM=XX+(AB(2)*XX+B(2))*RWD**2
  3       SUM1=SUM
          IF(BETA.NE.1.)SUM=SUM+(AB(N)*XX+B(N))*RWD**N
          IF(BETA.EQ.1.)THEN
             MM=2*N
             SUM=SUM+(AB(MM)*XX+B(MM))*RWD**MM
          ENDIF
          IF(DABS((SUM-SUM1)/SUM).LE.TOLR)THEN
            Y2=SUM
            GO TO 4
          ELSE
            IF(N.GE.60.OR.M.GE.60)THEN
               Y2=SUM
               PRINT *,'ITR LIM EXCEEDED FOR Y2'
               GO TO 4
          ELSE
            N=N+1
            GO TO 3
          ENDIF
          ENDIF

C         CALCULATION OF 'D'
  4       SUM=0.
          N=1
  5       SUM1=SUM
          IF(BETA.NE.1.)SUM=SUM+(N+1)*AB(N+1)*RWD**(N+1)
          IF(BETA.EQ.1.)THEN
             MM=2*N
             SUM=SUM+MM*AB(MM)*RWD**MM
          ENDIF
          IF(DABS((SUM-SUM1)/SUM).LE.TOLR)THEN
            D=-S*SUM
            GO TO 40
          ELSE
            IF(N.GE.59.OR.MM.GE.60)THEN
              D=-S*SUM
              PRINT *,'ITR LIM EXCEEDED FOR D'
              GO TO 40
          ELSE
            N=N+1
            GO TO 5
          ENDIF
          ENDIF

C         CALCULATION OF 'F'
  40      SUM=0.
          N=1
  50      SUM1=SUM
```

```
      IF(BETA.NE.1.)SUM=SUM+(N+1)*AB(N+1)+(N+2)*AB(N+2)
      IF(BETA.EQ.1.)THEN
         IF(N.EQ.1)MM=2*N
         IF(N.NE.1)MM=N+1
         SUM=SUM+MM*AB(MM)
      ENDIF
      IF(DABS((SUM-SUM1)/SUM).LE.TOLR)THEN
        F=SUM
        GO TO 6
      ELSE
        IF(N.GE.59)THEN
          F=SUM
          PRINT *,'ITR LIM EXCEEDED FOR F'
          GO TO 6
      ELSE
        N=N+2
        GO TO 50
      ENDIF
      ENDIF
C     CALCULATION OF 'G'
    6 SUM=1.
      N=1
    7 SUM1=SUM
      IF(BETA.NE.1.)SUM=SUM+N*B(N)+AB(N)+(N+1)*B(N+1)+AB(N+1)
      IF(BETA.EQ.1.)THEN
         J=2*N
         SUM=SUM+J*B(J)+AB(J)+(J+2)*B(J+2)+AB(J+2)
      ENDIF
      IF(DABS((SUM-SUM1)/SUM).LE.TOLR)THEN
        G=SUM
        GO TO 8
      ELSE
        IF(N.GE.60.OR.J.GE.60)THEN
          G=SUM
          PRINT *,'ITR LIM EXCEEDED FOR G'
          GO TO 8
      ELSE
        N=N+2
        GO TO 7
      ENDIF
      ENDIF
C     CALCULATION OF 'E'
    8 SUM=1.
      N=1
   11 SUM1=SUM
      IF(BETA.NE.1.)THEN
      SUM=SUM+(N*AB(N)*DLOG(RWD)+N*B(N)+AB(N))*RWD**N
      ELSE
        J=2*N
        SUM=SUM+(J*AB(J)*DLOG(RWD)+J*B(J)+AB(J))*RWD**J
      ENDIF
      IF(DABS((SUM-SUM1)/SUM).LE.TOLR)THEN
        E=-S*SUM
        GO TO 12
      ELSE
        IF(N.GE.60.OR.J.GE.60)THEN
          E=-S*SUM
          PRINT *,'ITR LIM EXCEEDED FOR E'
          GO TO 12
      ELSE
        N=N+1
        GO TO 11
      ENDIF
```

```
      ENDIF

C     CALCULATION OF 'C1' AND 'C2'
 12   AL=E*F-D*G
      C1=-G/AL
      C2=F/AL

C     CALCULATION OF THE TRANSFORMED SOLUTION (AT RD=RWD)

      PWDL=C1*Y1+C2*Y2
      RETURN
      END
************************************************************************
```

```
*                        PROGRAM # 5
*                        -----------
C                   THE STEHFEST ALGORITHM
C                *****************************
C
      FUNCTION PWD(TD,N)
C          THIS FUNTION COMPUTES NUMERICALLY THE LAPLACE TRNSFORM
C          INVERSE OF F(S).
      COMMON BETA,M,ICODE,JCODE
      IMPLICIT  REAL*8 (A-H,O-Z)
      DIMENSION G(50),V(50),H(25)
C
C          NOW IF THE ARRAY V(I) WAS COMPUTED BEFORE THE PROGRAM
C          GOES DIRECTLY TO THE END OF THE SUBRUTINE TO CALCULATE
C          F(S).
      IF (N.EQ.M) GO TO 17
      M=N
      DLOGTW=0.6931471805599
      NH=N/2
C
C          THE FACTORIALS OF 1 TO N ARE CALCULATED INTO ARRAY G.
      G(1)=1
      DO 1 I=2,N
      G(I)=G(I-1)*I
1     CONTINUE
C
C          TERMS WITH K ONLY ARE CALCULATED INTO ARRAY H.
      H(1)=2./G(NH-1)
      DO 6 I=2,NH
      FI=I
      IF(I-NH) 4,5,6
4     H(I)=FI**NH*G(2*I)/(G(NH-I)*G(I)*G(I-1))
      GO TO 6
5     H(I)=FI**NH*G(2*I)/(G(I)*G(I-1))
6     CONTINUE
C
C          THE TERMS (-1)**NH+1 ARE CALCULATED.
C          FIRST THE TERM FOR I=1
      SN=2*(NH-NH/2*2)-1
C
C          THE REST OF THE SN'S ARECALCULATED IN THE MAIN RUTINE.
C
C
C          THE ARRAY V(I) IS CALCULATED.
      DO 7 I=1,N
C
C          FIRST SET V(I)=0
      V(I)=0.
C
C          THE LIMITS FOR K ARE ESTABLISHED.
C          THE LOWER LIMIT IS K1=INTEG((I+1/2))
      K1=(I+1)/2
C
C          THE UPPER LIMIT IS K2=MIN(I,N/2)
      K2=I
      IF (K2-NH) 8,8,9
9     K2=NH
```

```
C
C           THE SUMMATION TERM IN V(I) IS CALCULATED.
8       DO 10 K=K1,K2
        IF (2*K-I) 12,13,12
12      IF (I-K) 11,14,11
11      V(I)=V(I)+H(K)/(G(I-K)*G(2*K-I))
        GO TO 10
13      V(I)=V(I)+H(K)/G(I-K)
        GO TO 10
14      V(I)=V(I)+H(K)/G(2*K-I)
10      CONTINUE
C
C           THE V(I) ARRAY IS FINALLY CALCULATED BY WEIGHTING
C           ACCORDING TO SN.
        V(I)=SN*V(I)
C
C           THE TERM SN CHANGES ITS SIGN EACH ITERATION.
        SN=-SN
7       CONTINUE
C
C           THE NUMERICAL APPROXIMATION IS CALCULATED.
17      PWD=0.
        A=DLOGTW/TD
        DO 15 I=1,N
        ARG=A*I
        PWD=PWD+V(I)*PWDL(ARG)
15      CONTINUE
        PWD=PWD*A
18      RETURN
        END
```

```
*                        PROGRAM # 6
*                        -----------
C                   THE STEHFEST ALGORITHM
C                ******************************
C
      FUNCTION PWD(TD,N)
C          THIS FUNTION COMPUTES NUMERICALLY THE LAPLACE TRNSFORM
C          INVERSE OF F(S).
      COMMON RWD,BETA,TOLR,M
      IMPLICIT  REAL*8 (A-H,O-Z)
      DIMENSION G(50),V(50),H(25)
C
C          NOW IF THE ARRAY V(I) WAS COMPUTED BEFORE THE PROGRAM
C          GOES DIRECTLY TO THE END OF THE SUBRUTINE TO CALCULATE
C          F(S).
      IF (N.EQ.M) GO TO 17
      M=N
      DLOGTW=0.6931471805599
      NH=N/2
C
C          THE FACTORIALS OF 1 TO N ARE CALCULATED INTO ARRAY G.
      G(1)=1
      DO 1 I=2,N
      G(I)=G(I-1)*I
1     CONTINUE
C
C          TERMS WITH K ONLY ARE CALCULATED INTO ARRAY H.
      H(1)=2./G(NH-1)
      DO 6 I=2,NH
      FI=I
      IF(I-NH) 4,5,6
4     H(I)=FI**NH*G(2*I)/(G(NH-I)*G(I)*G(I-1))
      GO TO 6
5     H(I)=FI**NH*G(2*I)/(G(I)*G(I-1))
6     CONTINUE
C
C          THE TERMS (-1)**NH+1 ARE CALCULATED.
C          FIRST THE TERM FOR I=1
      SN=2*(NH-NH/2*2)-1
C
C          THE REST OF THE SN'S ARECALCULATED IN THE MAIN RUTINE.
C
C
C          THE ARRAY V(I) IS CALCULATED.
      DO 7 I=1,N
C
C          FIRST SET V(I)=0
      V(I)=0.
C
C          THE LIMITS FOR K ARE ESTABLISHED.
C          THE LOWER LIMIT IS K1=INTEG((I+1/2))
      K1=(I+1)/2
C
C          THE UPPER LIMIT IS K2=MIN(I,N/2)
      K2=I
      IF (K2-NH) 8,8,9
9     K2=NH
C
C          THE SUMMATION TERM IN V(I) IS CALCULATED.
8     DO 10 K=K1,K2
```

```
      IF (2*K-I) 12,13,12
12    IF (I-K) 11,14,11
11    V(I)=V(I)+H(K)/(G(I-K)*G(2*K-I))
      GO TO 10
13    V(I)=V(I)+H(K)/G(I-K)
      GO TO 10
14    V(I)=V(I)+H(K)/G(2*K-I)
10    CONTINUE
C
C          THE V(I) ARRAY IS FINALLY CALCULATED BY WEIGHTING
C          ACCORDING TO SN.
      V(I)=SN*V(I)
C
C          THE TERM SN CHANGES ITS SIGN EACH ITERATION.
      SN=-SN
7     CONTINUE
C
C          THE NUMERICAL APPROXIMATION IS CALCULATED.
17    PWD=0.
      A=DLOGTW/TD
      DO 15 I=1,N
      ARG=A*I
      PWD=PWD+V(I)*PWDL(ARG)
15    CONTINUE
      PWD=PWD*A
18    RETURN
      END
```