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GYPSY FIELD PROJECT IN RESERVOIR CHARACTERIZATION

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Principal Investigator:	Daniel J. O'Meara Jr.
Project Manager	Robert E. Lemmon Bartlesville Project Office
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Submitted to:	Pittsburgh Energy Technology Center U.S. Department of Energy Wallace Road, Building 921 (Overnight Mail) P.O. Box 10940, MS 921-118 Pittsburgh, Pennsylvania 15236 (412) 892-4862 (Telephone) Document Control AD-20

100 East Boyd, Room 510, Norman, Oklahoma 73019
Telephone: (405) 325-3821 Fax: (405) 325-3180

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GYPSY FIELD PROJECT
IN RESERVOIR CHARACTERIZATION

Objectives

The overall objective of this project is to use the extensive Gypsy Field laboratory and data set as a focus for developing and testing reservoir characterization methods that are targeted at improved recovery of conventional oil.

The Gypsy Field laboratory, as described by Doyle, O'Meara, and Witterholt (1992), consists of coupled outcrop and subsurface sites which have been characterized to a degree of detail not possible in a production operation. Data from these sites entail geological descriptions, core measurements, well logs, vertical seismic surveys, a 3D seismic survey, crosswell seismic surveys, and pressure transient well tests.

The overall project consists of four interdisciplinary sub–projects which are closely interlinked:

1. Modeling depositional environments.
2. Sweep efficiency.
3. Tracer testing.
4. Integrated 3D seismic interpretation.

The first of these aims at improving our ability to model complex depositional environments which trap movable oil. The second is a development geophysics project which proposes to improve the quality of reservoir geological models through better use of 3D seismic data. The third investigates the usefulness of a new numerical technique for identifying unswept oil through rapid calculation of sweep efficiency in large reservoir models. The fourth explores what can be learned from tracer tests in complex depositional environments, particularly those which are fluvial dominated.

Summary of Technical Progress

During this quarter, the main activities involved the "Modeling depositional environments" Project", for which the progress is reported below:

1. Introduction. We study the determination of possibly discontinuous reservoir parameter functions defined on two dimensional regions from sparse pointwise measurements supplemented with measurements of a nonlinear function of the parameter. The specific application we have in mind is that of determining a permeability function from core measurements and pressure data, cf [6]. Our approach involves two steps. The first is to detect the discontinuous behavior, and the second is to isolate and refine the region containing it. For the first step we use a regularized output least squares procedure in which the reservoir mapping is approximated by linear combinations of bicubic B-splines. The regularization used is the H^1 seminorm that is related to the potential energy functional of an elastic membrane. This regularization gives sufficient compactness to obtain the existence of a solution to the associated minimization problem while implying minimal additional smoothing. Moreover, it seems to be well suited for the detection of the discontinuities and sudden changes so often exhibited by geological mappings [3,6]. Having as least detected an anomaly, we next attempt to isolate it by estimating its magnitude and a region containing it. The result of the procedure is to obtain a discontinuous function. We then essentially subtract this function from the model coefficient thereby, at least intuitively, reducing the discontinuous behavior. Again we consider the detection step to test for further discontinuous behavior. The procedure is repeated for further refinements.

In Section 2 we discuss the estimation of parameters by a regularized output least squares method in which there are data available in the form of measurements of permeability and pressure at locations within a two dimensional reservoir. Of interest here is the detection of the presence of features in the permeability function. The basic existence and approximation results are presented as well as resolution properties of the estimation procedure. If one uses a procedure for the purpose of detecting anomalies, the issue of the sensitivity of the method naturally arises and is discussed in Section 3. In Section 4 we describe a procedure to isolate the anomalies that have been detected. Essentially we seek a piecewise constant function possessing a rectangular subregion containing the feature of interest. Having found such a function, we again consider the regularized output least squares method to detect further anomalies in the permeability function repeating the procedure. Finally, Section 5 is devoted to the reporting the outcome of several numerical experiments.

2. Regularized Output Least Squares Estimation. We study the determination of a spatially dependent permeability mapping from measurements of the permeability at various locations along with pressure measurements. Towards this end, let Ω be an open domain in R^2 representing the reservoir with a Lipschitz boundary Γ. Let $K = K(x,y)$ be a real-valued function defined on Ω denote a permeability function that we wish to estimate. We suppose that measurements $\{K_i\}_{i=1}^{N_o}$ of K are available at N_o locations $\{x_i\}_{i=1}^{N_o}$ along with measurements $\{z_i\}_{i=1}^{N_o}$ of pressure p. We assume that fluid is injected at x_1. While at x_{N_o}, the condition $p(x_{N_o}) = 0$ holds. The pressure p is a function of K according to Darcy's law

$$-\nabla \cdot (K \nabla p) = f \text{ in } \Omega, \tag{2.1}$$

with the boundary condition

and define the set

$$Q_{ad} = \{K \in Q'_{ad} : \langle \phi_i, K\rangle = K^{(i)} \text{ for } i = 1, ..., N_o\}.$$

The linear functionals ϕ_i may be of the type described above.

The approach we use to recover K from the data obtained from measurements $\{K_i\}_{i=1}^{N_o}$ of K and measurements $\{z_i\}_{i=1}^{N_o}$ of p is the so-called regularized output least squares method [7]. Hence, we formulate the following minimization problem

(2.5) Find $K_0 \in Q_{ad}$ such that $J(K_0) = \text{minimum } J(K)$ subject to $K \in Q_{ad}$

where

$$J(K) = \int_\Omega |\nabla K|^2 \, dx \; + \; \gamma \sum_{i=1}^{N_o} (\langle \Phi_i, p(K)\rangle - z_i)^2$$

thereby accomodating the pressure measurements. The linear functional constraints may be included by penalization. Hence, in the penalized case setting

$$L(K) \; = \; J(K) + \frac{1}{\epsilon} \sum_{i=1}^{N_o} (\langle \phi_i, K\rangle - K_i)^2 \tag{2.6}$$

we have

$$\text{minimize } L(K) \tag{2.7}$$

$$\text{subject to } K \in Q'_{ad}$$

In [7] it is proved that the mapping $K \mapsto p(K)$ is continuous from Q_{ad} with the weak $H^1(\Omega)$ topology into $H^1(\Omega)$ with the weak topology, see also [2]. Existence of a solution follows immediately.

Proposition 2.1. For each ϵ, γ, μ, and ν greater than 0, there exist solutions to problems (2.5) and (2.7). As ϵ approaches zero, weak cluster points $H^1(\Omega)$ of the solutions K_ϵ of problem (2.7) are solutions of (2.5).

Because the mapping $K \mapsto p(K)$ is nonlinear, problem (2.7) is solved numerically by iteration. For example, starting with an initial guess for K, the procedure seeks to update K by means of a descent method. In each step it is necessary to solve a system that approximates (2.1)-(2.3) as well as a gradient system. The approximating system is obtained by the finite element method. Let $\{\alpha_i\}_i^N$ and $\{\psi_i\}_i^M$ be sets of N and M linearly independent functions in $H^1(\Omega)$, respectively. Express p and K as linear combinations

$$p = \sum_{i=1}^{N} c_i \alpha_i \tag{2.8}$$

and

$$K = \sum_{i=1}^{M} a_i \psi_i. \tag{2.9}$$

Define the matrices for $k = 1, ..., M$

$$(G^{(k)})_{ij} = \int_\Omega \psi_k \nabla \alpha_i \cdot \nabla \alpha_j \mathrm{dx} \tag{2.10}$$

and $i, j = 1, ..., N$, as well as the column-vector ρ of length N with k-th component

$$\rho_k = \int_\Omega f \alpha_k \mathrm{dx}. \tag{2.11}$$

The discrete analogue of problem (2.1)-(2.3) is given by

$$\sum_{k=1}^{M} a_k G^{(k)} c = \rho. \tag{2.12}$$

with the additional condition

$$\sum_{k=1}^{M} c_k \langle \Phi_{N_o}, \alpha_k \rangle = 0. \tag{2.13}$$

Based on the minimization problem (2.4), we may obtain boundary value problems in which the supplementary condition (2.13) is included by means of a penalization. In this case equations (2.1)-(2.3) are replaced by the boundary value problem

$$-\nabla \cdot (K \nabla p) \ + \ \frac{1}{\eta} \langle \Phi_{N_o}, p \rangle \ \Phi_{N_o} = f \text{ in } \Omega \tag{2.14}$$

$$\frac{\partial p}{\partial n} = 0 \text{ on } \Gamma \tag{2.15}$$

where $\frac{1}{\eta}$ serves as a penalization of the constraint $\langle \Phi_{N_o}, p \rangle = 0$ as $\eta \to 0$. Thus, introducing the N x N matrix G_ϕ defined by

$$(G_\phi)_{kj} = \langle \Phi_{N_o}, \alpha_k \rangle \langle \Phi_{N_o}, \alpha_j \rangle$$

and setting

$$G = \sum_{k=1}^{M} a_k G^{(k)} + \frac{1}{\eta} G_\phi,$$

equations (2.14) and (2.15) are replaced by the equation

$$G\,c = \rho \tag{2.16}$$

Therefore, given a vector $a = (a_1, ..., a_M)$ (the function K), we may calculate the vector $c = (c_1, ..., c_N)$ (the function p) as the solution to equation (2.16). With these values we may now evaluate the functional $L(a) = L(K)$. The solution of (2.6) is that vector a_0 minimizing the functional $L(a)$ subject to the conditions specifying Q'_{ad}. Define the matrices and vectors

$$(H)_{ij} = \sum_{k=1}^{N_o} \langle \Phi_k, \alpha_i \rangle \langle \Phi_k, \alpha_j \rangle \tag{2.17}$$

$$\zeta_i = \sum_{k=1}^{N_o} \langle \Phi_k, \alpha_i \rangle z_i \tag{2.18}$$

$$C_z = \sum_{k=1}^{N_o} z_i^2$$

for $i, j = 1, ..., N$, and for $i, j = 1, ..., M$,

$$(G_0)_{ij} = \int_\Omega \nabla \psi_i \cdot \nabla \psi_j \mathrm{dx} \tag{2.19}$$

$$(H_K)_{ij} = \sum_{n=1}^{N_o} \langle \phi_n, \psi_i \rangle \langle \phi_n, \psi_j \rangle \tag{2.20}$$

$$\kappa_i = \sum_{n=1}^{N_o} \langle \phi_n, \psi_i \rangle K^{(n)} \tag{2.21}$$

$$C_K = \sum_{k=1}^{N_o} (K^{(n)})^2 \tag{2.22}$$

The functional L(a) takes the form

$$L(a) = \gamma\{c(a)^* H c(a) - 2\zeta^* c(a) + C_z\} + a^*(G_0 + \frac{1}{\epsilon} H_K) a + \frac{1}{\epsilon}(-2\kappa^* a + C_K) \tag{2.23}$$

where $c(a)$ is the solution of (2.16) associated with the parameter vector a. The minimization problem is given by

$$\text{minimize } L(a) \tag{2.24}$$

$$\text{subject to } K \in Q'_{ad}.$$

The minimization problem (2.24) is solved as an unconstrained problem assuming essentially that the solution strictly satisfies the inequality constraints specifying Q'_{ad}. This is reasonable from a physical point of view since the bounds μ and ν are posed primarily for the mathematical formulation of the problem. Numerically, a steepest descent method is used decrease the value of $L(a)$ iteratively. In carrying out this procedure it is necessary to compute the gradient of $L(a)$. The derivative of $L(a)$ with increment $h = col(h_1, ..., h_M)$ is given by

$$DL(a)h = 2\{\gamma(Hc(a) - \zeta)^* Dc(a)h \ + \ ((G_0 + \frac{1}{\epsilon}H_K)a \ - \ \frac{\kappa}{\epsilon})^* h\} \tag{2.25}$$

where the derivative of $c(a)$ with increment h, $Dc(a)h$, is the solution of the equation

$$GDc(a)h = -\sum_{l=1}^{M} h_l G^{(l)} c(a) \tag{2.26}$$

The derivative of $L(a)$ and the derivative of $c(a)$ are an M-vector and an N x M matrix, respectively.

By introducing the equation

$$G\pi = Hc(a) - \zeta, \tag{2.27}$$

we observe that

$$\pi^* GDc(a)h = (Hc(a) - \zeta)^* Dc(a)h.$$

Hence, it follows that

$$(Hc(a) - \zeta)^* Dc(a)h = -\sum_{k=1}^{M} h_k(\pi^* G^{(k)} c(a))$$

and (2.25) may be expressed as

$$\frac{1}{2}DL(a)h = -\gamma \sum_{k=1}^{M} h_k(\pi^* G^{(k)} c(a)) + \{(G_o + \frac{1}{\epsilon}H_K) - \frac{1}{\epsilon}\kappa\}^* h.$$

Defining the column M-vector

$$\Xi = [\pi^* G^{(k)} c(a)]_{k=1}^{M},$$

we may write $DL(a)h$ as follows

$$\frac{1}{2}DL(a)h = \{-\gamma\Xi + (G_o + \frac{1}{\epsilon}H_K)a - \frac{\kappa}{\epsilon}\}^* h \tag{2.28}$$

and

$$DL(a) = 2[-\gamma\Xi + (G_o + \frac{1}{\epsilon}H_K)a - \frac{\kappa}{\epsilon}].$$

The optimization algorithm starts with an initial guess $a^{(0)}$. On the k-th iteration with the parameter vector $a^{(k)}$ the procedure is as follows

Compute $c(a^{(k)})$ as the solution of (2.16)

Compute $L(a^{(k)})$ by (2.23)

Compute π by solving (2.27)

Compute $DL(a)$ by (2.28)

Update $a^{(k)}$to find $a^{(k+1)}$ according to the criterion $a^{(k+1)} = a^{(k)} - \beta DL(a^{(k)})$

where β is chosen so that

$$L(a^{(k+1)}) \leq L(a^{(k)}).$$

Finally, we give the optimality system characterizing solutions of (2.24) that belong to the interior of Q_{ad}. These solutions necessarily satisfy the condition $DL(a)h = 0$ for any M-vector h. From equation (2.28) we conclude that

$$(G_o + \frac{1}{\epsilon}H_K)a - \frac{\kappa}{\epsilon} - \gamma\Xi = 0 \tag{2.29}$$

The optimality system is recorded in the following.

Proposition 2.2. The optimality conditions satisfied by interior solutions of (2.24) are given by

$$Gc(a) = \rho$$

$$G\pi = Hc(a) - \zeta$$

$$(G_o + \frac{1}{\epsilon}H_K)a - \frac{\kappa}{\epsilon} - \gamma\Xi = 0$$

where

$$\Xi = [\pi^* G^{(k)} c(a)]_{k=1}^{M},$$

3. Uniqueness and Sensitivity In this section we obtain conditions assuring there is at most one interior solution. We will then see that these conditions also imply the diffentiability of the solution with respect to the data. The starting point is the system comprising the optimality conditions given in Proposition 2.2. We assume there two sets of data distinguished by sub(super)scripts 1 and 2. Defining the vectors

$$d = c_1 - c_2$$

$$\delta = \pi_2 - \pi_1$$

$$\alpha = a_2 - a_1$$

$$\zeta = \zeta_2 - \zeta_1$$

and

$$\kappa = \kappa_2 - \kappa_1,$$

we obtain the equations

$$G_2 d = -\sum_{k=1}^{M} \alpha_k G^{(k)} c_1$$

$$G_2 \delta = -\zeta - \sum_{k=1}^{M} \alpha_k G^{(k)} \pi_1 + Hd$$

$$(G_o + \frac{1}{\epsilon} H_K)\alpha = \frac{\kappa}{\epsilon} + \gamma\{\delta^* G^{(l)} c_2\}_{l=1}^{M} +$$

$$+\gamma\{\pi_1^* G^{(l)} d\}_{l=1}^{M}$$

from the optimality system. To determine estimates, suppose there are positive constants ν_0, ν, K_0, and K_1 such that

(3.1) $$d^* G_i d \geq \nu \mid d \mid^2,$$

for $i = 1, 2$

(3.2) $$K_0 \geq \mid G^{(k)} \mid,$$

for $k = 1, ..., M,$

(3.3) $$K_1 \geq \mid H \mid,$$

and

(3.4) $$d^*(G_o + \frac{1}{\epsilon} H_K) d \geq \nu_0 \mid d \mid^2 .$$

The conditions (3.1) hold because of the positive definitness of the matrix on the left side in the underlying model. Condition (3.4) is a consequence of the presence of H_K. With these assignments, we obtain the following estimates for $i = 1, 2$

$$\mid c_i \mid \leq \frac{\mid \rho \mid}{\nu}, \tag{3.5}$$

and thus

$$\mid G_i^{-1} \mid \leq \frac{1}{\nu},$$

$$\mid d \mid \leq \frac{K_0}{\nu^2} \mid \rho \mid \mid \alpha \mid, \tag{3.6}$$

$$\mid \delta \mid \leq \frac{K_0}{\nu^2}\left(\frac{2K_1}{\nu} \mid \rho \mid + \mid \zeta_1 \mid\right) \mid \alpha \mid \ + \frac{\mid \zeta \mid}{\nu}, \tag{3.7}$$

and

$$\nu_0 \mid \alpha \mid \leq \gamma K_0 \frac{\mid \rho \mid}{\nu} \mid \delta \mid + \tag{3.8}$$

$$+\gamma K_0\left(\frac{K_1}{\nu^2} \mid \rho \mid + \frac{\mid \zeta_1 \mid}{\nu}\right) \mid d \mid + \frac{\mid \kappa \mid}{\epsilon}.$$

It follows that

$$\nu_0 \mid \alpha \mid \leq \frac{\mid \kappa \mid}{\epsilon} \ + \ \frac{\gamma K_0}{\nu^2} \mid \rho \mid \mid \zeta \mid \ + \ \frac{\gamma {K_0}^2}{\nu^3} \mid \rho \mid \left(\frac{3K_1}{\nu} \mid \rho \mid \ + \ 2 \mid \zeta_1 \mid\right) \mid \alpha \mid. \tag{3.9}$$

Set

$$\beta = \nu_0 - \frac{\gamma {K_0}^2}{\nu^3}\left(\frac{3K_1}{\nu} \mid \rho \mid \ + \ 2 \mid \zeta_1 \mid\right) \mid \rho \mid. \tag{3.10}$$

Proposition 3.1. If β given by (3.10) is positive where the constants are assigned according to (3.1)-(3.4), then

$$\mid \alpha \mid \leq \frac{1}{\beta}\left(\frac{1}{\epsilon} \mid \kappa \mid \ + \ \frac{K_0}{\nu^2} \mid \rho \mid \mid \zeta \mid\right).$$

If $\zeta_1 = \zeta_2$ and $\kappa_1 = \kappa_2$, then uniqueness follows.

Corollary 3.2. If $\beta > 0$, then there is at most one solution a belonging to int Q_{ad}.
Remark 3.3 Observe that the condition $\beta > 0$ may be satisfied by choosing the vector ρ associated with the forcing term to be sufficiently small.

In order to detect changes in the optimal parameter obtained by means of the regularized output least squares technique in response to changes in the data, it is useful to examine the differentiability of the optimal estimator with respect to the data. Hence, we next consider conditions under which the mapping associating an optimal estimator a with a data vector ζ is differentiable. Let us suppose then that $a \in Q = \mathcal{R}^M$ and ρ and $\zeta \in X = \mathcal{R}^N$. In fact we suppose that a is a regularized output least squares solution in the interior of Q_{ad}. Thus, define the function $a \mapsto G(a)$ from $\mathcal{R}^M$ to the linear space of N x N real-valued matrices by

$$G(a) = \sum_{k=1}^{M} a_k G^{(k)},$$

and set

$$G = G(a) + \frac{1}{\eta} G_\phi.$$

We assume that a is associated with the system of equations given in Proposition 2.2. To examine the differentiability of a with respect to ζ, define the mapping $F : Q \text{ x } X \mapsto X$ by

$$F(a,\zeta) = (G_o + \frac{1}{\epsilon} H_K)a - \frac{\kappa}{\epsilon} - \gamma\{\pi(a,\zeta)^* G^{(k)} c\}_{k=1}^{M}.$$

By the implicit function theorem [1] the function $\zeta \mapsto a(\zeta)$ is differentiable with respect to ζ if the Frechet partial derivatives of F with respect to a and ζ, $D_a F(a,\zeta)$ and $D_\zeta F(a,\zeta)$ exist and the linear operator $D_a F(a,\zeta) : Q \mapsto Q$ is nonsingular.

To calculate the required derivatives, it is necessary to calculate the derivative of each of the functions $a \mapsto c(a)$ and $(a,\zeta) \mapsto \pi(a,\zeta)$. Setting $G(h) = \sum_{k=1}^{M} h_k G^{(k)}$, we note that

$$GDc(a)h = -G(h)c(a)$$

so that

$$Dc(a)h = -G^{-1}G(h)c(a).$$

Further,

$$GD_a\pi(a,\zeta)(h) = -G(h)\pi(a,\zeta) \ + \ HDc(a)h,$$

and from Proposition 2.2,

$$D_a\pi(a,\zeta)(h) = -G^{-1}[G(h)G^{-1}H \ + \ HG^{-1}G(h)]c(a) \ + \ G^{-1}G(h)G^{-1}\zeta.$$

Hence, we may write

$$Dc(a)h = -\sum_{i=1}^{M} h_i G^{-1} G^{(i)} c(a),$$

and

$$D_a\pi(a,\zeta)h = \sum_{i=1}^{M} h_i\{G^{-1}G^{(i)}G^{-1}\zeta \ - \ G^{-1}[G^{(i)}G^{-1}H \ + \ HG^{-1}G^{(i)}]c(a) \ \}$$

On the other hand the Frechet partial derivative with respect to a is given by

$$D_aF(a,\zeta)h = (G_o + \frac{1}{\epsilon}H_K)h \ - \{(D_a\pi(a,\zeta)h)^*G^{(k)}c \ + \ \pi^*G^{(k)}(Dc(a)h)\}_{k=1}^{M}.$$

Defining the M x M matrix

$$\mathcal{K}_{ki} = \{G^{-1}G^{(i)}\zeta - G^{-1}(G^{(i)}G^{-1}H \ + \ HG^{-1}G^{(i)})c\}^*G^{(k)}c \ - \ \pi^*G^{(k)}G^{-1}G^{(i)}c,$$

then we may write

$$D_aF(a,\zeta)h = (G_o + \frac{1}{\epsilon}H_K - \gamma\mathcal{K})h.$$

To calculate the partial derivative of F with respect to ζ with increment δ, we first note that

$$D_\zeta\pi(a,\zeta)\delta = -G^{-1}\delta.$$

It follows that the partial derivative with respect to ζ with increment δ is given by

$$D_\zeta F(a,\zeta)\delta = \{c^*G^{(k)}G^{-1}\delta\}_{k=1}^{M} \tag{3.11}$$

Defining the M x N matrix $\mathcal{H}_\zeta$ in which the k-th row is given by $c^*G^{(k)}G^{-1}$, we have

$$D_\zeta F(a,\zeta)\delta = \mathcal{H}_\zeta\delta.$$

With these observations, we have the following.

Proposition 3.4. If the matrix $G_o + \frac{1}{\epsilon}H_K - \mathcal{K}$ is nonsingular, then

$$Da(\zeta)\delta = -D_aF(a,\zeta)^{-1}D_\zeta F(a,\zeta)\delta. \tag{3.12}$$

Remark 3.5. Observe that for given $G_o + \frac{1}{\epsilon}H_K$, the nonsingularity of $G_o + \frac{1}{\epsilon}H_K - \mathcal{K}$ may be controlled by the forcing vector ρ.

By applying the estimates previously used, we may obtain sufficient conditions for differentiability. Towards this end, observe from (3.4) that

$$|\,(G_o + \frac{1}{\epsilon}H_K)^{-1}\,| \leq \frac{1}{\nu_0}.$$

Thus, it follows by estimating the terms in the expression for $\mathcal{K}_{ki}$ that

$$|\,\mathcal{K}_{ki}\,| \leq (\frac{2{K_0}^2}{\nu^3}\,|\,\zeta\,||\,\rho\,| \ + \ \frac{3{K_0}^2K_1}{\nu^4}\,|\,\rho\,|^2\,.$$

We have the following.

Proposition 3.6. If

$$\gamma\frac{{K_0}^2}{\nu^3}(\frac{3K_1}{\nu} \mid \rho \mid \; + 2 \mid \zeta \mid) \mid \rho \mid < \nu_0,$$

then $Da(\zeta)$ exists.

Proof. Under the hypothesis, it follows a norm of $\gamma(G_o \; + \; \frac{1}{\epsilon}H_K)^{-1}\mathcal{K}$ is less than one. Hence,

$$(I \; - \; \gamma(G_o \; + \; \frac{1}{\epsilon}H_K)^{-1}\mathcal{K})^{-1}$$

exists and

$$G_o \; + \; \frac{1}{\epsilon}H_K - \gamma\mathcal{K}$$

is invertible. This implies that $D_aF(a,\zeta)$ is invertible. Since $Da(\zeta)$ exists, the differentiability of a with respect to ζ follows.

Remark 3.7. The conditions of Propositions 3.1 and 3.5 are the same. Hence, it follows that under (3.1) that there is at most one solution and the function $\zeta \mapsto a(\zeta)$ is differentiable.

Remark 3.8. The differentiability of the mapping $\zeta \mapsto a(\zeta)$ provides a tool with which we can investigate the sensitivity of the interior optimal estimators with respect to perturbations in the data ζ. We shall develop these ideas further in a later paper. However, we note that for a perturbation δ of the N-column data vector ζ

$$a(\zeta + \delta) = a(\zeta) + Da(\zeta)\delta + o(\mid \delta \mid).$$

Hence, if the M x N matrix $Da(\zeta)$ has a nontrivial null space $\mathcal{N}$, then it is not possible to detect the perturbation $\delta \in \mathcal{N}$ as a first order effect. Define the N-vectors $\theta_k = c^*G^{(k)}G^{-1}$ for $k = 1, ..., M$ From (3.11) and (3.12), we see that $\mathcal{N}$ is determined as the orthogonal complement of the span of the vectors $\{\theta_k\}_{k=1}^{M}$. It follows, for example, that if M ¡ N there always are vectors δ that are not detectable.

4. Estimation of the Discontinuity. Having detected a discontinuity, our next step is to isolate and obtain some estimate of it. We proceed by considering an example in which the admissible permeability functions K have a discontinuity determined by two regions within Ω parameterized by 2 real numbers a and b. We denote these two regions by $\Omega(a,b)$ and $\Omega \setminus \Omega(a,b)$. An admissible permeability K is also parameterized constants K_1 and K_2 modelling the magnitude of the discontinuity between the regions. Hence, K takes the form

$$K(x,y) = K_0(x,y) + K_e(x,y) \tag{4.1}$$

where

$$K_e(x,y) = K_1 \text{ if } (x,y) \in \Omega(a,b), \text{ and } K_2 \text{ otherwise.}$$

It is assumed that the function K_0 is known. Introducing the characteristic function Ξ of the set $\Omega(a,b)$, we may write

$$K_e(x,y) = (K_1 - K_2)\Xi(x,y) \; + \; K_2. \tag{4.2}$$

and equations (4.1) and (4.2) may be thought of as replacing equation (2.9). To fix ideas let us suppose that $\Omega(a,b)$ is a rectangle of the form $(x_0 - a, x_0 + a)\mathrm{x}(y_0 - b, y_0 + b)$. The stiffness matrix is now given as

$$(G)_{ij} = \int_\Omega (K_0 + K_2)\nabla\alpha_i \cdot \nabla\alpha_j dxdy + (K_1 - K_2)\int_{x_0-a}^{x_0+a}\int_{y_0-b}^{y_0+b} \nabla\alpha_i \cdot \nabla\alpha_j dxdy$$

Setting

$$(G_0)_{ij} = \int_\Omega \nabla\alpha_i \cdot \nabla\alpha_j dxdy,$$

$$(G_1)_{ij} = \int_\Omega K_0\nabla\alpha_i \cdot \nabla\alpha_j dxdy,$$

and

$$(G_2)_{ij}(a,b) = \int_{x_0-a}^{x_0+a}\int_{y_0-b}^{y_0+b} \nabla\alpha_i \cdot \nabla\alpha_j dxdy, \tag{4.3}$$

we have

$$G = G_1 + (K_1 - K_2)G_2(a,b) + K_2 G_0. \tag{4.4}$$

Thus, we obtain an equation analogous to (2.16) given by

$$Gc = \rho. \tag{4.5}$$

with the approximating solution u expressed as $u = \sum_{i=1}^{N} c_i\alpha_i$.

Define the criterion

$$J(a,b,K_1,K_2) = \sum_{j=1}^{N_o}(\langle\Phi_j, u\rangle \; - \; z_j)^2$$

and the functional

$$\mathcal{N}(a,b,K_1,K_2) \; = \; \sum_{j=1}^{N_K}(\langle\phi_j, K\rangle \; - \; K_j)^2.$$

Under discretization, the functional J takes the form

$$J(a,b,K_1,K_2) = c^* H_P c - 2\zeta^* c + c_z$$

Note that $\mathcal{N}(a,b,K_1,K_2)$ is a discontinuous function and is, in fact, a piecewise constant function of a and b. Hence, we look for a, b, K_1, and K_2 that minimizes J while keeping $\mathcal{N}(a,b,K_1,K_2)$ at its minimum value, call it $\mathcal{N}_0$. That is, we seek

$$a,\ b,\ K_1,\ \text{and}\ K_2\ \text{minimizing}\ J(a,b,K_1,K_2)\ \text{subject to}\ \mathcal{N}=\mathcal{N}_0.$$

To carryout this program we calculate the partial derivatives of J and c. We find that $d_1 = D_{K_1}c$ and $d_2 = D_{K_2}c$ satisfy the equations

$$Gd_1 = G_2(a,b)c \tag{4.6}$$

and

$$Gd_2 = (G_0 - G_2(a,b))c, \tag{4.7}$$

respectively. To compute $d_a = D_a c$ and $d_b = D_b c$, it is necessary to differentiate the integral in (4.3). Observe that the partial derivatives of the matrix G_2 are matrices whose entries are given by

$$(D_aG_2)_{ij} = \int_{y_0-b}^{y_0+b} \{(\nabla\alpha_i \cdot \nabla\alpha_j)(x_0+a,y) - (\nabla\alpha_i \cdot \nabla\alpha_j)(x_0-a,y)\}dy$$

and

$$(D_bG_2)_{ij} = \int_{x_0-a}^{x_0+a} \{(\nabla\alpha_i \cdot \nabla\alpha_j)(x,y_0+b) - (\nabla\alpha_i \cdot \nabla\alpha_j)(x,y_0-b)\}dx.$$

Thus, we find that d_a and d_b must satisfy the equations

$$Gd_a = -(K_1-K_2)D_aG_2c$$

and

$$Gd_b = -(K_1-K_2)D_bG_2c,$$

respectively. The partial derivatives of J now are calculated by the chain rule.

5. A numerical example. We consider a problem in which we specify a coefficient K(x,y) and generate pressure data based on that function by solving the problem (2.1)-(2.3)for p with a specific forcing function f by finite elements. Using this data we then attempt to recover K. Let $\Omega = (0,1)\text{x}(0,1)$ and suppose that measurements of pressure and permeability can be made at locations (0.175,0.175), (0.835,0.175), (0.5,0.5), (0.175,0.835), and (0.835,0.835). For a test permeability function we use the following

$$K_{test}(x,y) = \begin{cases} 8+3.5cos(x+y)\ \text{for}\ (x,y)\in(0,0.3)\text{x}(0,0.3), \\ 4+2.5cos(x+y-2),\ \text{for}(x,y)\in(0.75,1)\text{x}(0.75,1), \\ 2+cos(x+y),\ \text{otherwise} \end{cases}.$$

shown in Figure 1. Further, we suppose that $p = 0$ at the point (0.835,0.835) and that fluid is injected at the point (0.175,0.175). The resulting pressure function obtain by means of a finite element solution is portrayed in Figure 2. For the approximations to the pressure, we use tensor products of cubic B-splines [5] defined on a uniform mesh determined by subdividing (0,1) into seven subintervals. Since imposing Neumann boundary conditions improves accuracy, we use 64 basis functions for approximating pressure adjusted to incorporate the Neumann boundary condition. For approximating the parameter, we again use tensor products of cubic B-splines but defined on a mesh determined by subdividing (0,1) into 5 equal subintervals. Imposing no boundary conditions, we then use 64 basis functions to approximate the parameter. Using data at the observation points, we apply the regularized output least squares method as a detection procedure resulting in Figure 3. Based on this result, we search for a coefficient of the form

$$K_1(x,y) = \begin{cases} K_1 if (x,y) \in (0,a) \text{ x } (0,b), \\ 2 \text{ otherwise} \end{cases}$$

using the technique discussed in the previous section. The result is illustrated in Figure 4. We then apply further detection by again using the regularized output least squares method to estimate the coefficient K_2 where the permeability has the form

$$K(x,y) = K_1(x,y) \ + \ K_2(x,y).$$

The result is portrayed in Figure 5. Based on this computation, we again use the procedure for discontinuous coefficients by searching for discontinuities based on the rectangle $(b,1)\text{x}(b,1)$. This yields the Figure 6. Again, we use the regularized output least square method to look for further discontinuities and background with the outcome given in Figure 7. The resulting pressure from the estimated coefficient is given in Figure 8.

6. Conclusions. We have formulated the output least square estimation technique and have discussed its use as a detection tool for problems with pressure data by giving conditions for uniqueness and differentiability of optimal estimated permeability function with respect to perturbations of the data. In addition we introduced a method to estimate the location and magnitude of a jump discontinuity. We also presented a numerical example for the location of discontinuities in a permeability function in the presence of a background. By alternating detection and discontinuity estimation procedures, it seems to be possible to construct coefficients with discontinuities in the presence of a background function.

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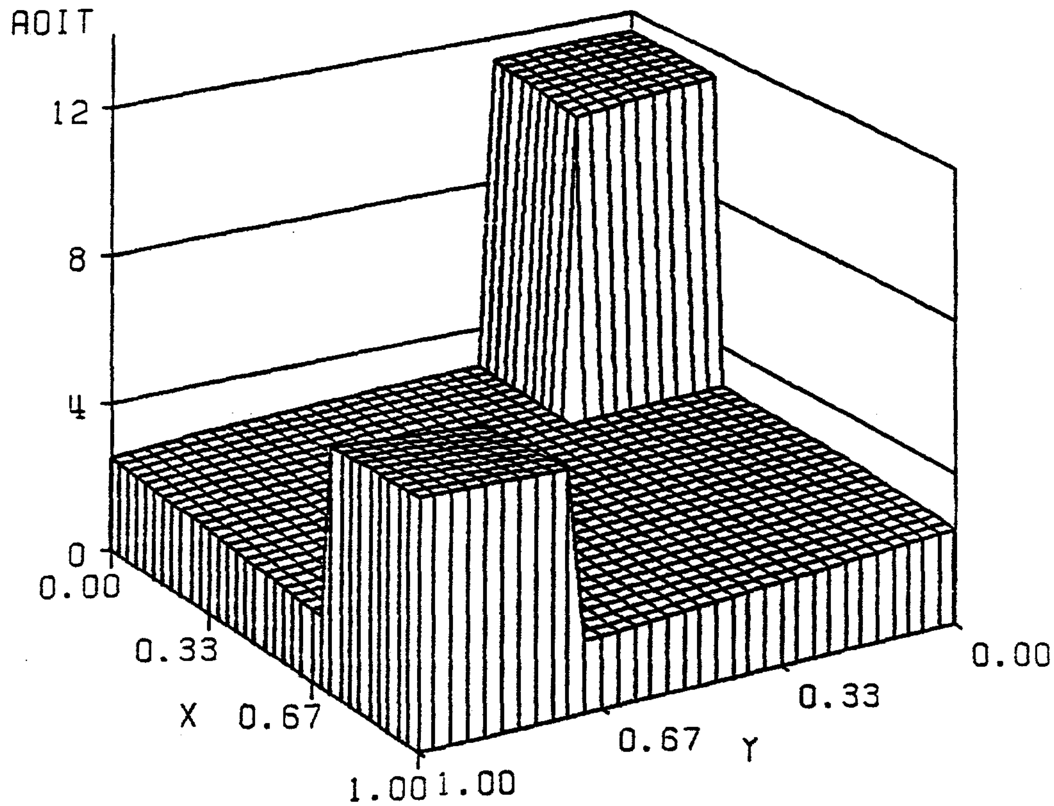
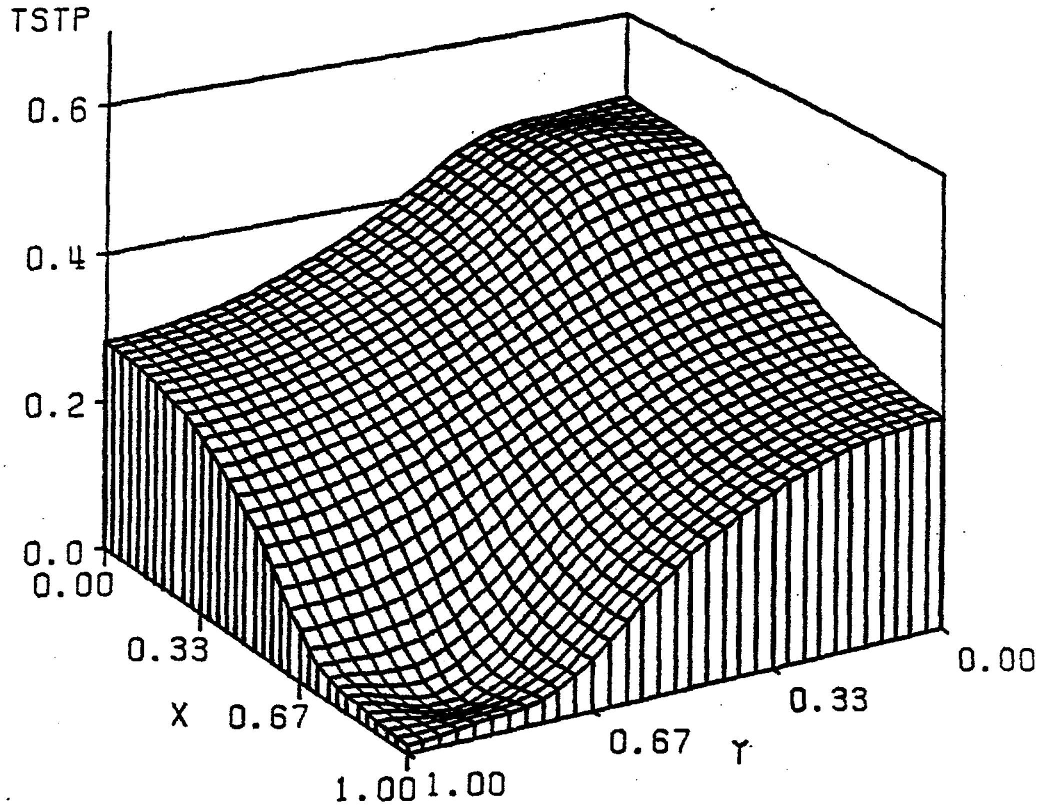


STEP I --- A = AO (REG.)

Figure 1



STEP I --- A = AO (REG.)

Figure 2

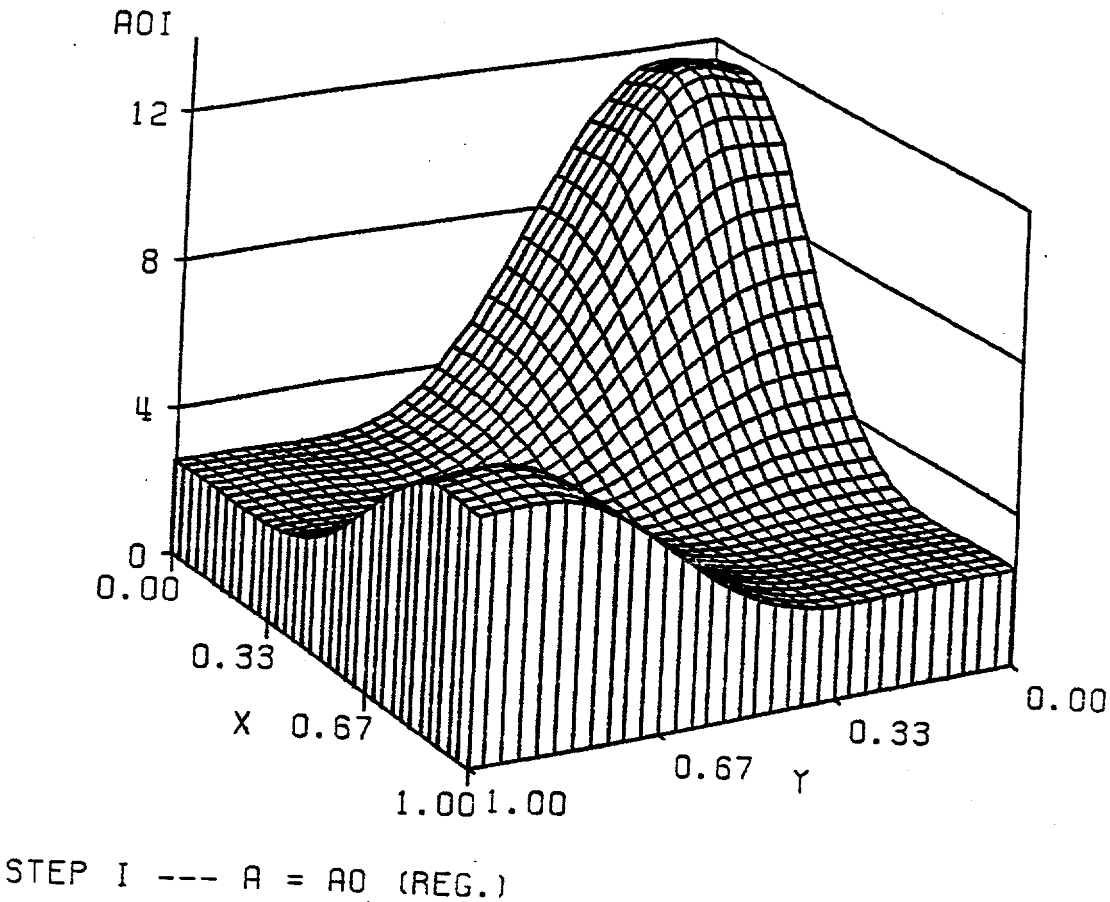
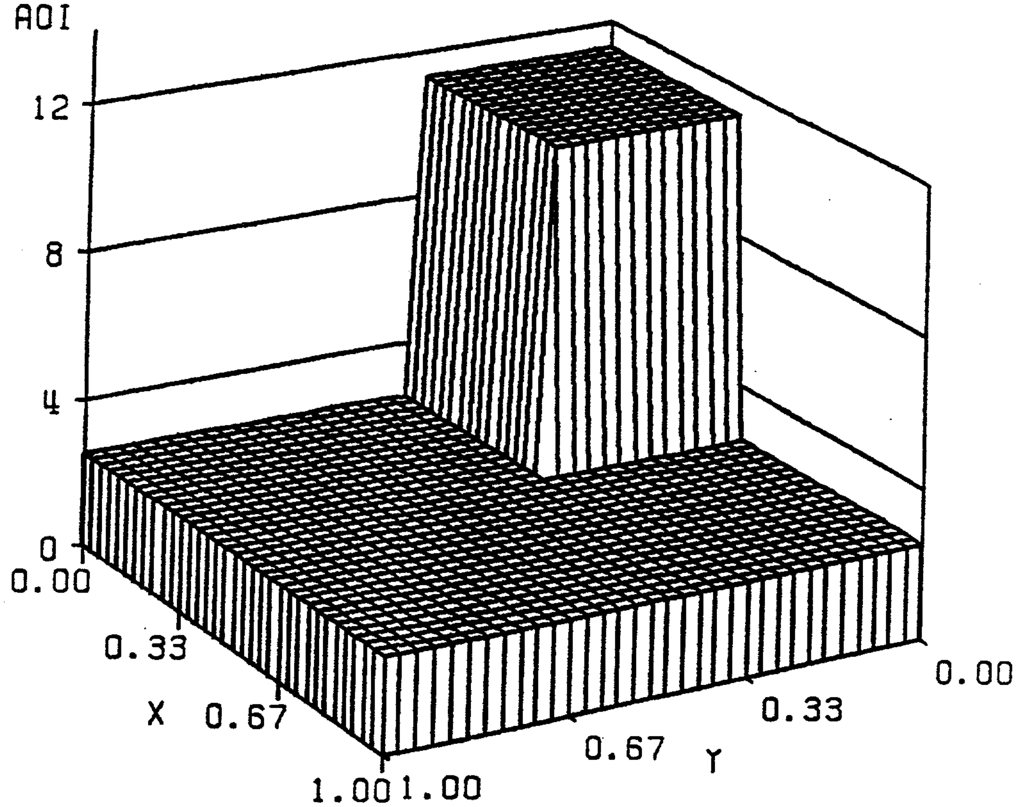
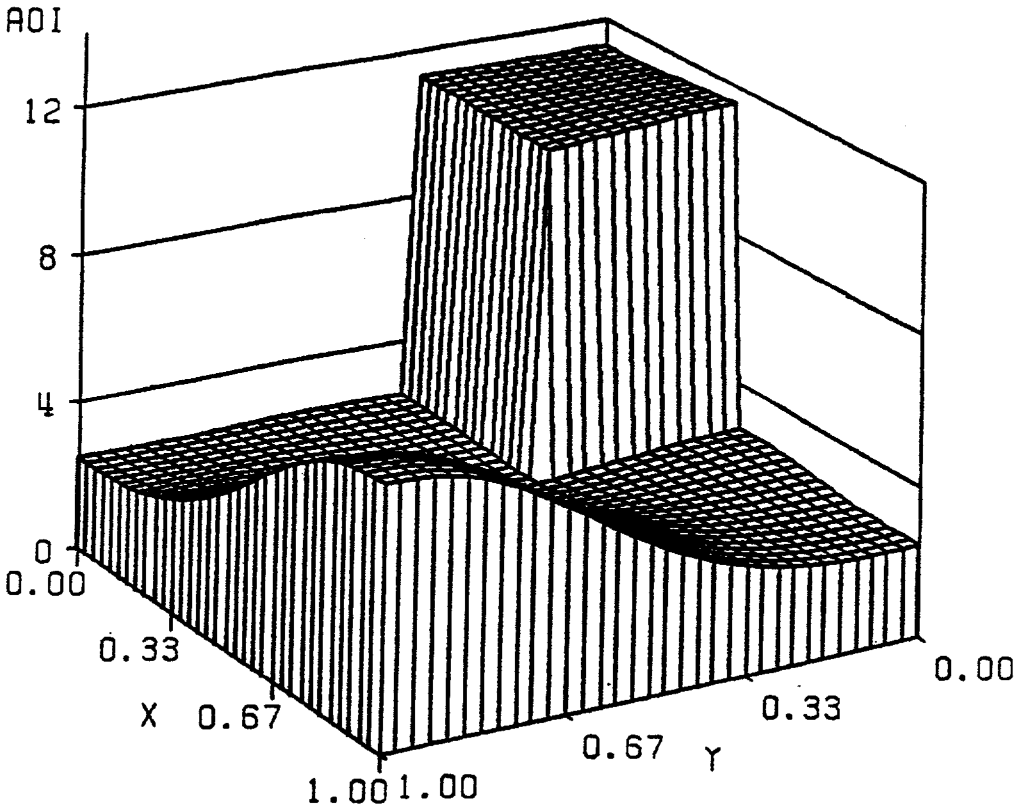


Figure 3



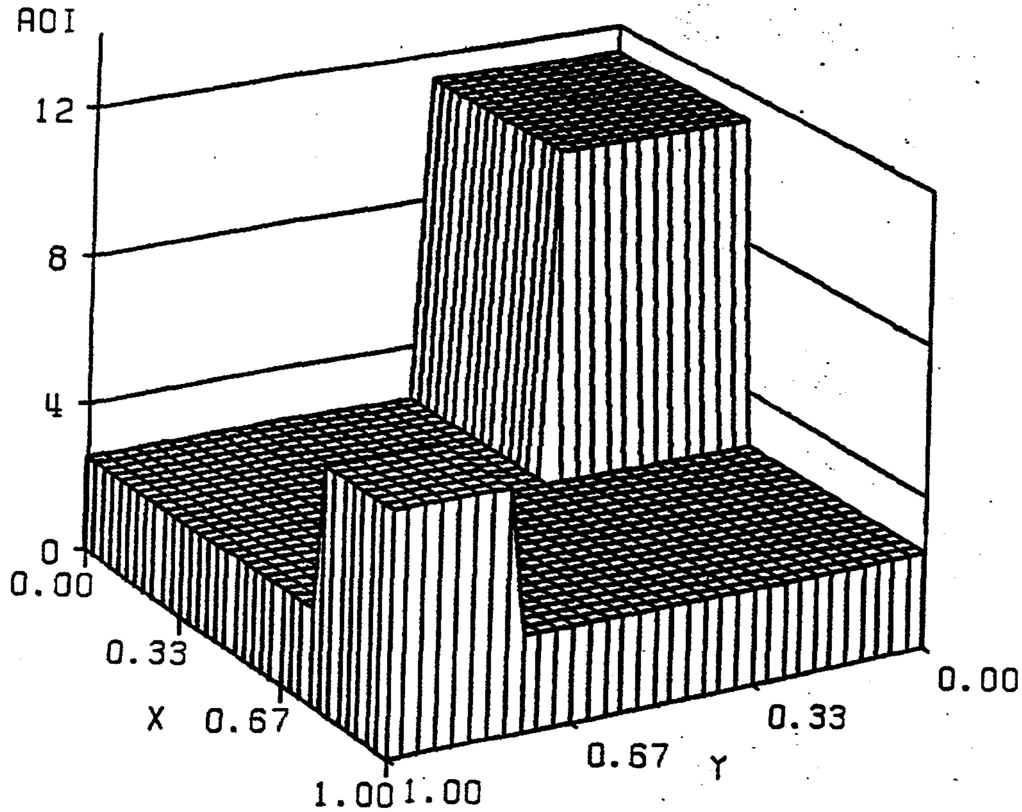
STEP I+1/2------ A = AO(DISC.)

Figure 4



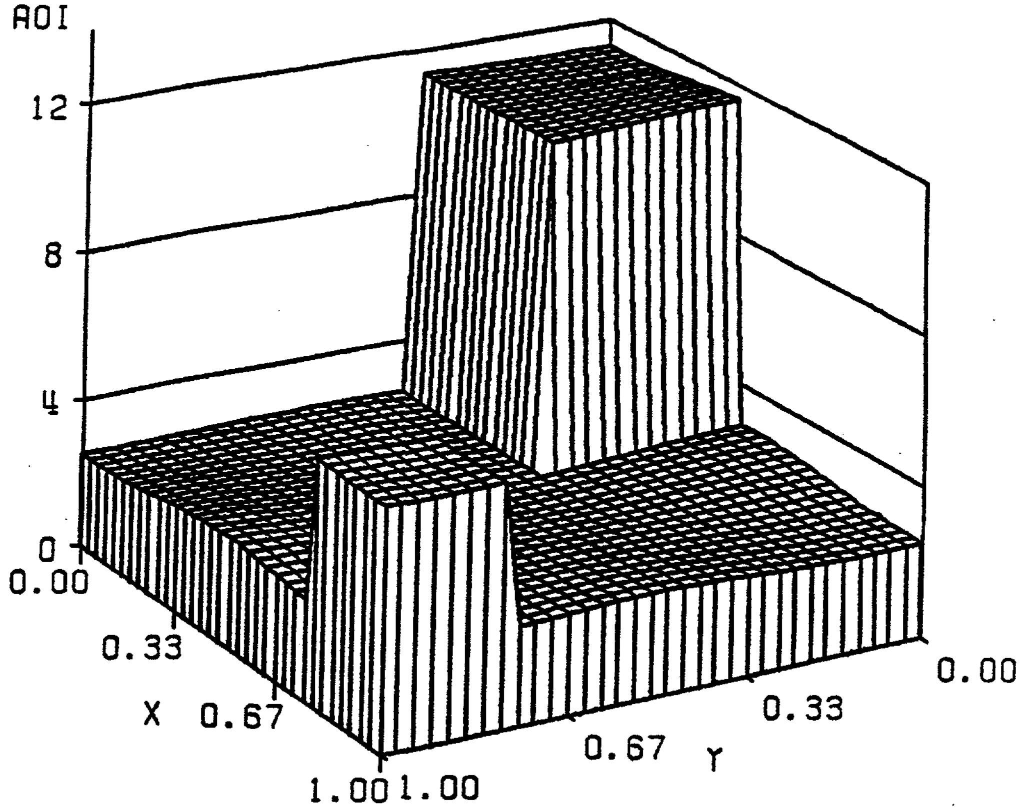
STEP II --- A = A0(DISC.) + A1 (REG.)

Figure 5



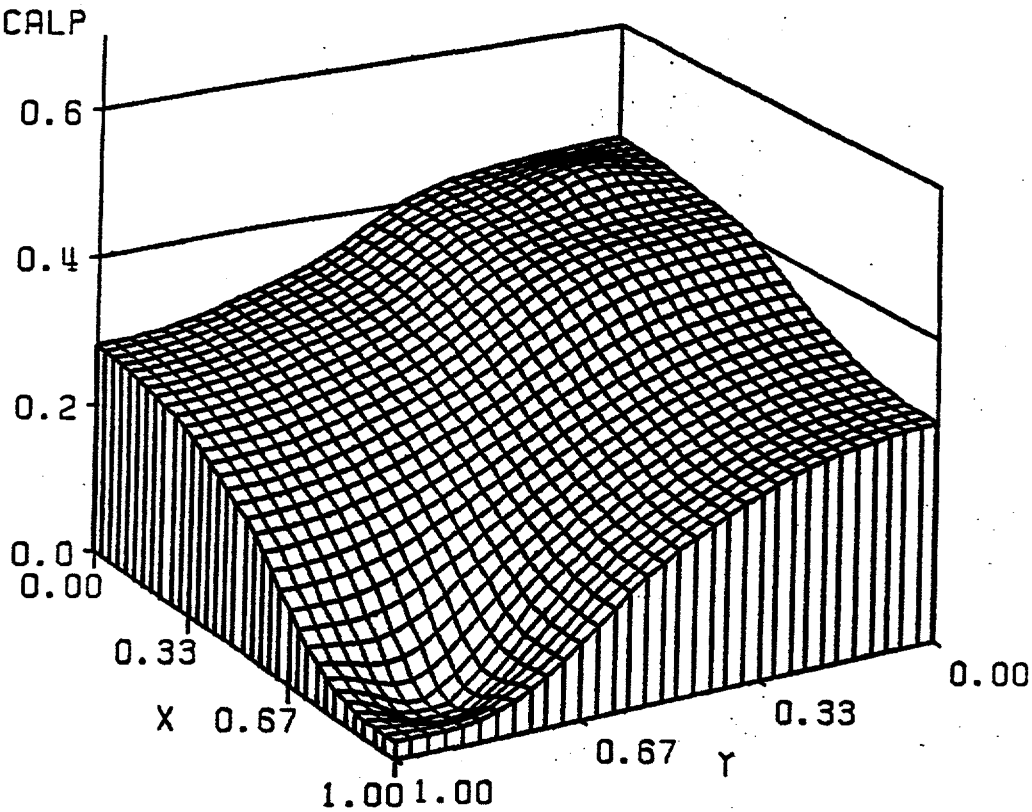
STEP III --- A=A0(DISC.)+A1(DISC.)

Figure 6



STEP IV --- A=AOD+A1D+A2R

Figure 7



STEP IV --- A=A0D+A1D+A2R

Figure 8