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Doublets and Other Allied Well Patterns

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By
William E. Brigham

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Prepared for
U.S. Department of Energy
Assistant Secretary for Fossil Energy

Thomas Reid, Project Manager
National Petroleum Technology Office
P.O. Box 3628
Tulsa, OK 74101

Prepared by
Stanford University
Department of Petroleum Engineering
Green Earth Sciences Bldg., Room 080b
367 Panama Street
Stanford, CA 94305-2220

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ACKNOWLEDGEMENTS

Let me start these acknowledgments with a disclaimer. When first reading these notes, you may get the idea that almost all the material herein has been developed by me; and only by me. This is not true!

Many of the ideas can be found in Muskat's excellent book, "Flow of Homogeneous Fluids Through Porous Media." I have referenced Muskat occasionally, but I often found his equation developments to be quite brief and difficult to follow. So I developed them my own way -- and, of course, ended up with the same equations.

One pioneer in this type of work was H.J. Morel-Seytoux of Colorado State University. He has written a number of remarkably good papers and notes on isolated patterns and their behavior. Unfortunately they are scattered through the literature. He also put them together for a reservoir engineering course he taught at Chevron. Unfortunately, most people will not have access to these notes, so I have not referenced him herein. My excuse is, "Why reference something that will be difficult to find?" But his work has been very good indeed.

I've also referred to Craft and Hawkins' excellent book, "Applied Petroleum Reservoir Engineering." In particular, I've pointed out some errors in that book. As a result, the casual reader may get the impression that I think the book is not very good. Far from it! It is an excellent text. But, however good a book is, it is almost impossible to write it without having some errors. I have found, to my dismay, that it is even hard to write an eight page paper without an error in it--so it is not surprising that a full book has some. Knowing this, I'm sure some errors will be found here -- and I apologize in advance for them.

I know that I really should have included several dozen references here. Why haven't I? I can make the (possibly) valid argument that the goal was to get the important ideas across and not bore the reader with these less crucial matters. To be honest, however, the idea of the many hours that would have been necessary to look up all the appropriate references was so daunting that I decided not to do it. I hope the people who read this find it readable, interesting and enlightening. If you do, I'll feel the effort was worthwhile.

This work was supported by the Assistant Secretary for Fossil Energy, Office of Oil Gas and Shale Technologies of the U. S. Department of Energy under Contract Numbers DE-FC26-00BC15311 and DE-FG22-96BC14994 to Stanford University. Likewise, the support of the SUPRI-A Industrial Affiliates is gratefully acknowledged.

INTRODUCTION

Whenever a liquid is injected into an infinite reservoir containing liquid with the same flow properties, the equations of flow are well known. The pressures in such a system vary over time and distance (radius) in ways that depend on the formation and liquid flow properties. Such equations are well known -- they form the basis for the voluminous well-testing literature in petroleum engineering and ground water hydrology.

In all these equations, the early time behavior depends on the characteristics in and near the injection well. Later in time, assuming no leakage from the formation, all these equations approach the well known log approximation, where the dimensionless pressure varies in a way that is proportional to the log of a dimensionless time term which includes the square of the radial distance away from the well. This log approximation is valid sooner near the active wells, and requires a longer time at distances further away from them.

Suppose there are two wells -- one an injector and one a producer -- with identical rates. The behavior of this system can be calculated using superposition; which merely means that the results can be added independently of each other. When this is done, the remarkable result is that after a period of time there is a region that approaches steady state flow. Thereafter, the pressures and flow velocities in this region stay constant. The size of this region increases with time.

This "steady state" characteristic can be used to solve a number of interesting and useful problems, both in heat transfer and in fluid flow. The heat transfer problems can be addressed because the equations are identical in form. A number of such problems are solved herein for doublet systems. In addition, concepts are presented to help solve other cases that flow logically from the problems solved herein.

It is not necessary that only two wells be involved. It turns out that any time the total injection and production are equal, the system approaches steady state. This idea is also addressed in these notes. A number of useful multiwell cases are addressed to present the flavor of such solutions. Various isolated patterns give interesting results that can be analyzed exactly, and further, can be logically compared with each other.

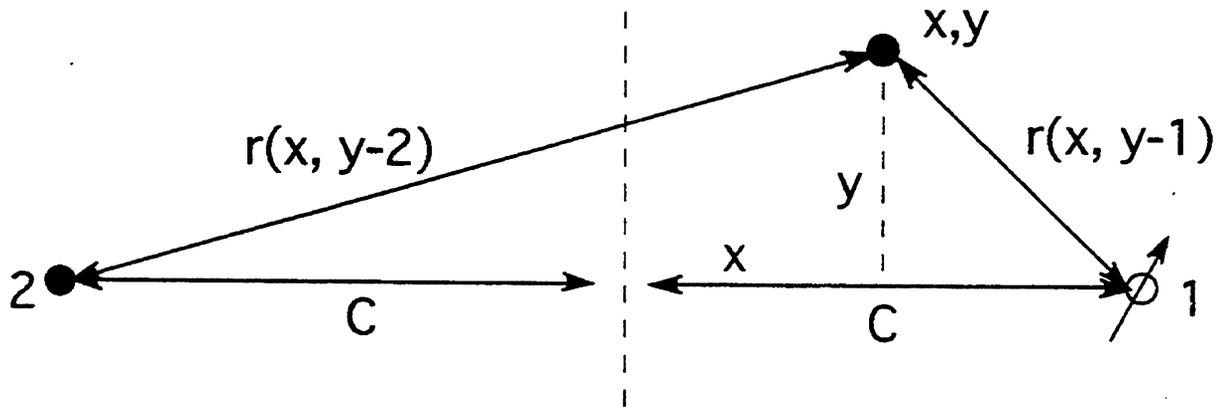
The nomenclature and equations used here are expressed in common petroleum engineering terms. However, it should be clear that these ideas are equally valid for problems in ground water hydrology. In fact, in general, they are more applicable to groundwater problems, for the assumption of unity mobility ratio is almost exactly correct for groundwater problems, while it is often not true in petroleum engineering problems.

1. SHAPES OF DOUBLET CONSTANT PRESSURE LINES AND FLOW LINES

In Muscat's book, Flow of Homogeneous Fluids p. 173, can be seen the constant pressure and flow lines around an injection/production well doublet pair. A reproduction of this figure (doublet to show the entire pressure/flow field) is attached on the next page. This figure appears to be made up of interlocking circles of differing radii with differing locations for their centers. It seems worthwhile to find whether the constant pressure lines are circles, and if so, to define their equations, and to perform the same analysis on the flow lines. This is the goal of these notes.

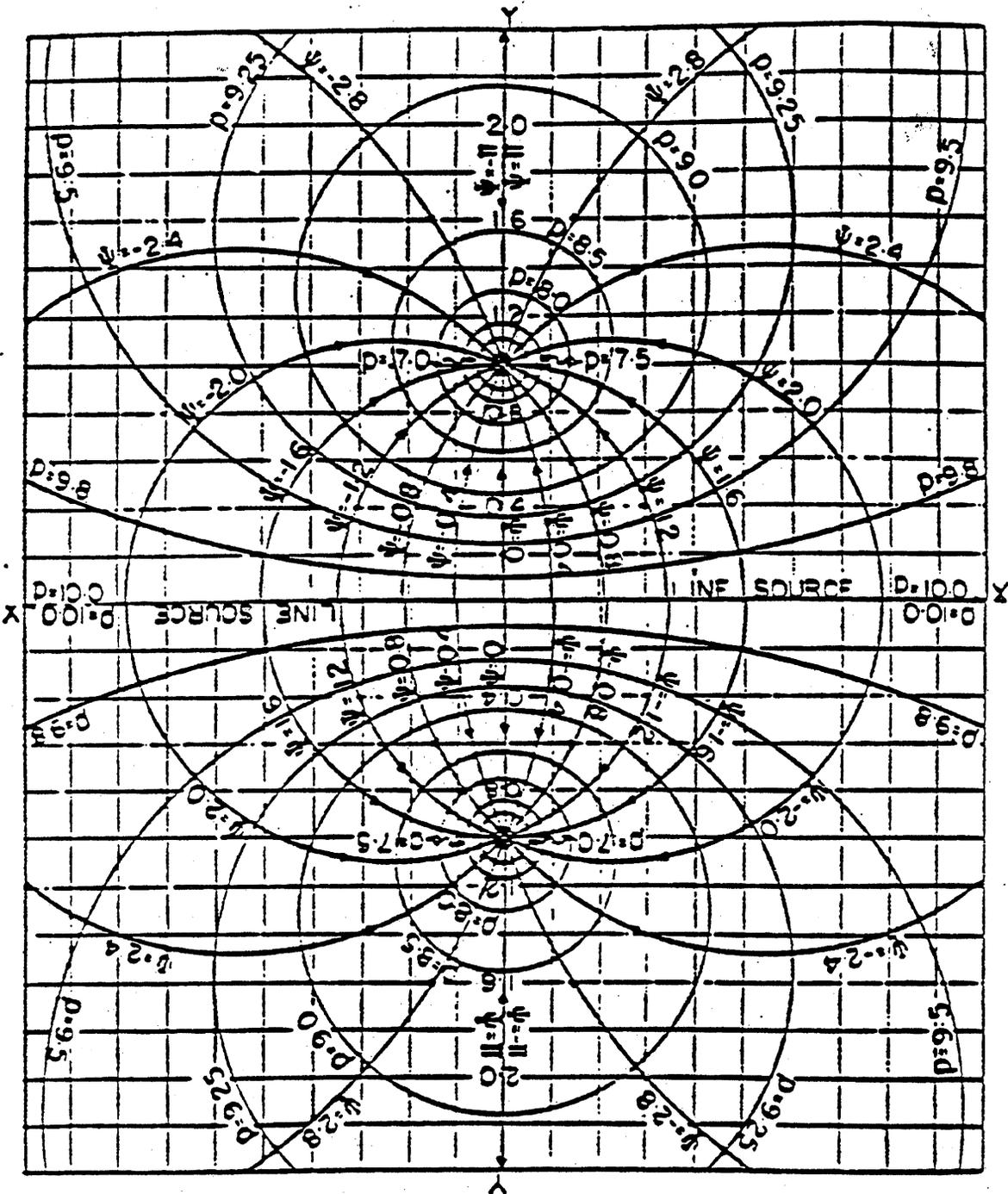
1.1 CONSTANT PRESSURE LINES

Let us consider a doublet system as shown on the sketch below.



A few words about this sketch would be appropriate. The vertical dashed line in the center goes through the origin to orient us to the coordinate system. Well 1 is an injector at distance C along the x axis from the origin; and similarly Well 2 is a producer along the negative x axis at distance C . The two wells are assumed to have the same rates, $q_1 = -q_2$. Because these two wells are equally spaced away from the origin, and because their rates are the same, the vertical dashed line through the origin always remains at the original pressure, p_i .

The point x, y is any general point in the right half (positive x) plate. We will later assume that this general point is at a fixed pressure and find the equation for the locus of all points at the same pressure.



In the sketch, the symbols, $r(x, y-1)$ and $r(x, y-2)$, may be a bit confusing. The term $r(x, y-1)$ is meant to show the radial distance from point x, y to Well 1, and similarly $r(x, y-2)$ is the radial distance from x, y to Well 2. I refer to radial distance here because the general solutions for pressure drop are in terms of radial distance as follows:

$$p_D(x, y) = -Ei \left[-\frac{t_D}{[r_D(x, y-1)]^2} \right] + Ei \left[-\frac{t_D}{[r_D(x, y-2)]^2} \right] \quad (1)$$

In Eq. 1, p_D , t_D and r_D can be put into dimensional terms, and after a period of time the exponential integrals (Ei) can be written with the logarithmic approximation. These substitutions result in the following equation.

$$\frac{2\pi kh[p(x, y) - p_i]}{q\mu} = \frac{1}{2} \left[\ln \left(\frac{kt}{\phi \mu c_t [r(x, y)-1]^2} \right) - \ln \left(\frac{kt}{\phi \mu c_t [r(x, y)-2]^2} \right) \right] \quad (2a)$$

Since the terms, k, t, ϕ, μ , and c_t are the same in both log terms, Eq. 2a simplifies to,

$$\frac{2\pi kh[p(x, y) - p_i]}{q\mu} = \frac{1}{2} \left[\ln[r(x, y)-2]^2 - \ln[r(x, y)-1]^2 \right] \quad (2b)$$

or, on simplifying further,

$$\frac{4\pi kh[p(x, y) - p_i]}{q\mu} = \ln \left[\frac{r(x, y-2)}{r(x, y-1)} \right]^2 = K' \quad (3)$$

Equation 3 is a simplification of Eq. 2b, but in addition it is shown to be a constant, K' . This is in recognition that we are restraining the locus of these general radii to be at a constant pressure. We need now to find the equation of the locus of these constant pressures.

We first note that if the logarithm of this ratio is constant, then the ratio itself is a constant, which we will call K , and then we merely need to define these radii ratios from the geometry of the system.

$$\frac{[r(x, y-1)]^2}{[r(x, y-2)]^2} = \frac{(C-x)^2 + y^2}{(C+x)^2 + y^2} = K \quad (4)$$

It is important to notice that the r ratio used in Eq. 4 has been inverted. This is for convenience. As defined, K' could range from 0 to ∞ , while K , as defined here ranges from 0 to 1. When $K' = \infty$, $K = 0$ and when $K' = 0$, $K = 1$.

We can now cross multiply Eq. 4, and collect like terms. The result is,

$$x^2(1-K) - 2Cx(1+K) + C^2(1-K) + y^2(1-K) = 0 \quad (5a)$$

or

$$x^2 - 2C \frac{x(1+K)}{(1-K)} + C^2 + y^2 = 0 \quad (5b)$$

The terms containing x on the left side can be made into a perfect square by adding the term, $C^2(1+K)^2/(1-K)^2$ to both sides of the equation. The result is,

$$x^2 - 2C \frac{x(1+K)}{(1-K)} + \frac{C^2(1+K)^2}{(1-K)^2} + y^2 = \frac{C^2(1+K)^2}{(1-K)^2} - C^2 \quad (6a)$$

or

$$\left[x - \frac{C(1+K)}{(1-K)} \right]^2 + y^2 = \frac{4C^2K}{(1-K)^2} \quad (6b)$$

Equation 6b is the general equation of a circle. Thus it clearly states that any constant pressure locus, that is, any constant K , produces a circle. Further, this equation tells us how these circles behave at differing pressures (differing K 's). First we see that all the circles have their centers along the x axis where $y = 0$. We should have expected this, for the wells are on the x axis and thus the solution should be vertically symmetrical.

Second, we see that the locations of the centers of the circles will depend on the value of the pressure, that is the value of K , as follows

$$x_c = \frac{C(1+K)}{(1-K)} \quad (7)$$

where x_c = Location of the center of any constant pressure circle

From Eq. 4 we see that K approaches 1.0 when the ratio of the radii approaches 1.0. This happens when K' in Eq. 3 approaches zero; that is, when $p(x, y)$ approaches p_i . From Eq. 7, at this condition, x_c approaches infinity. Thus this condition approaches an infinite circle with its left side at $x = 0$, at pressure equal to p_i .

At the other limit, K approaches zero, and K' approaches infinity. This condition defines the pressure near the injection well.

The third important aspect of Eq. 6b is that the right hand side defines the radius of each constant pressure circle, as follows

$$r_c^2 = \frac{4C^2K}{(1-K)^2} \quad (8)$$

where r_c = Radius of circle defined by Eq. 6(b)

Again, the results are logical and consistent. As K approaches 1.0, the radius approaches infinity, and as K approaches zero the radius becomes smaller, approaching the point source at Well 1. This second idea can be carried further by looking at the equation at the injection well where $r_{wi} \ll C, x \approx C$, and $y \approx 0$. From Eq. 4, we can write

$$K \approx \frac{r_{wi}^2}{(2C)^2} \ll 1 \quad (9)$$

Equation 8 then simplifies as follows

$$r_c^2 = \frac{4C^2K}{(1-K)^2} \approx 4C^2 \left(\frac{r_{wi}}{2C} \right)^2 = r_{wi}^2 \quad (10)$$

and Eq. 7 simplifies to

$$x_c = C \frac{(1+K)}{(1-K)} = C \quad (11)$$

Thus the equation for the constant pressure locus at the injection wellbore simplifies to the equation of the wellbore circle,

$$(x - C)^2 + y^2 = r_{wi}^2 \quad (12)$$

just as we would expect.

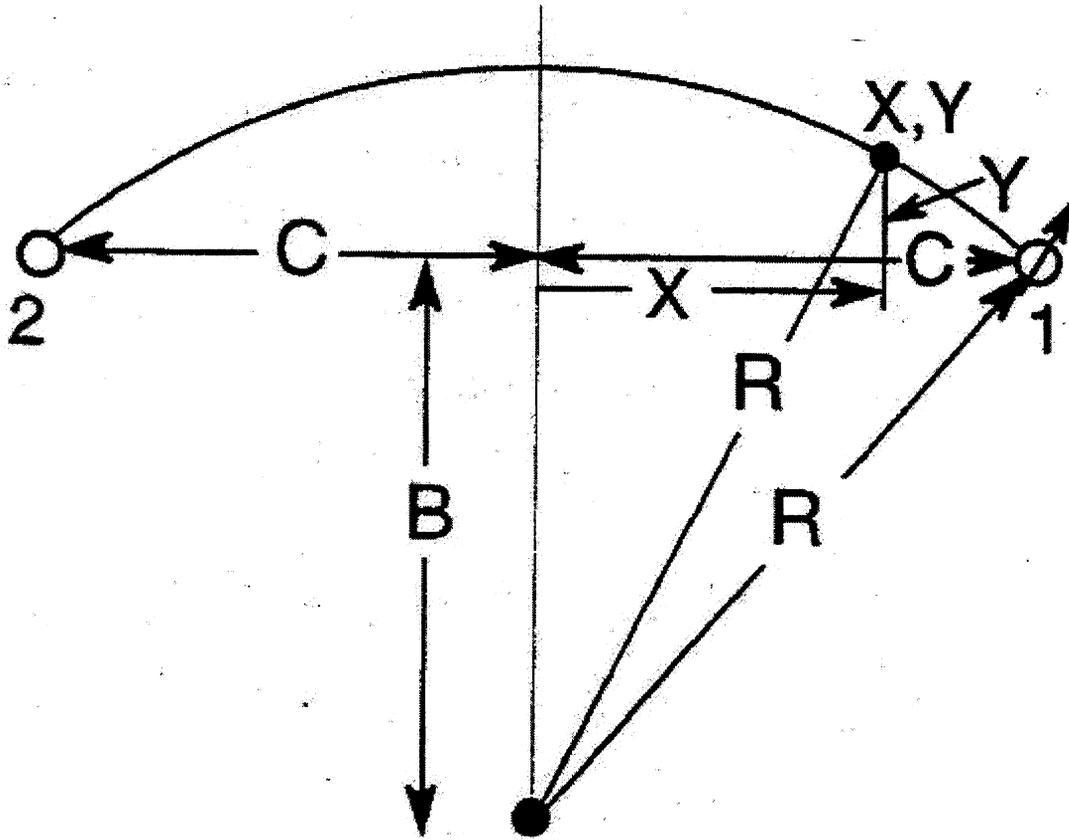
Since there are so many equations here, it may be that the story they are telling us has been lost in the mathematics. So let's summarize what these equations are saying. First, at the injection well the pressure is high at a circle representing the wellbore, Eq. 12. Away from the injection well, the pressures drop, and the shapes of the constant pressure loci are still circles as defined by Eq. 6b.

As the pressure drops, these circles get larger, and their centers move to the right of the well. Equation 6b defines the locations of these centers and the radii of the circles. They are functions of K which in turn is a function of the pressure. Note that the radii of the circles, from Eq. 8, are always smaller than the values of x for their centers, Eq. 7. Thus the circles always remain in the positive x half-plane. As the pressure in these circles approaches p_i , the initial pressure of the system, the circle becomes infinite in size with the left edge at $x = 0$.

No mention has been made of the picture of the left half-plane of this system. It seems obvious that no mention is needed; the left side will be an exact mirror image of the right side, with the pressures lower than p_i rather than higher.

1.2 FLOW LINES

Since we were right in assuming the constant pressure lines were circles, it seems likely the flow lines also are circles. To find out if they are, we will draw a general circle through the well doublet system, and then determine whether that circle is a flow line. A sketch of this system is on the following page.



A few words about this sketch would be appropriate. A general circle of radius, R , has been drawn through Wells 1 and 2 and through point x, y (three points define a circle). The circle must be symmetrical around the y axis since the wells are equally spaced. The location of the center of the circle is indicated along the y axis at $-B$.

If this circle is a flow line, the pressure gradient everywhere along the circle must be tangent to it. This is the only condition required to prove the circle is a flow line.

In general, the velocity vectors can be written as follows,

$$\frac{u_x}{u_y} = \frac{-(\partial p / \partial x)}{-(\partial p / \partial y)} = \frac{(\partial p / \partial x)}{(\partial p / \partial y)} \quad (13)$$

The perpendicular vector to the circle at x, y can be related to the velocity vector as follows,

$$\frac{u_x}{u_y} = \frac{-(B + y)}{x} \quad (14)$$

If the circles are flow lines, then Eq. 14 is correct. If they are not, then the equality does not exist. Equation 14 arises from the realization that in general the slope of any perpendicular vector is the negative inverse of the slope of the

original vector. In Eq. 14, $B + y$ is proportional to the vertical component, and x is proportional to the horizontal component. The method we will use is to define a general velocity vector equation on the circle using Eq. 13, and then find whether that equation satisfies Eq. 14. If it does, then the circle we assumed is correct.

We start first with the general equation for the pressure in our doublet field.

$$\frac{4\pi kh[p(x,y) - p_i]}{q\mu} = -\ln[(C-x)^2 + y^2] + \ln[(C+x)^2 + y^2] \quad (15)$$

We can differentiate Eq. 15 with respect to x and also with respect to y , as follows,

$$\begin{aligned} \frac{4\pi kh}{q\mu} \left(\frac{\partial p}{\partial x} \right) &= \frac{-2(C-x)}{(C-x)^2 + y^2} - \frac{2(C+x)}{(C+x)^2 + y^2} \\ &= \frac{-2(C-x)(C+x)^2 - 2y^2(C-x) - 2(C+x)(C-x)^2 - 2y^2(C+x)}{[(C-x)^2 + y^2][(C+x)^2 + y^2]} \end{aligned} \quad (16)$$

and

$$\begin{aligned} \frac{4\pi kh}{q\mu} \left(\frac{\partial p}{\partial y} \right) &= \frac{2y}{(C-x)^2 + y^2} - \frac{2y}{(C+x)^2 + y^2} \\ &= \frac{2y(C+x)^2 - 2y(C-x)^2}{[(C-x)^2 + y^2][(C+x)^2 + y^2]} \end{aligned} \quad (17)$$

Equation 16 can be divided by Eq. 17 to find the ratio of the pressure gradients,

$$\begin{aligned} \frac{\partial p / \partial x}{\partial p / \partial y} &= -\frac{(C-x)(C+x)^2 + (C-x)y^2 + (C+x)(C-x)^2 + (C+x)y^2}{y(C+x)^2 - y(C-x)^2} \\ &= -\frac{(C-x)(C^2 + 2Cx + x^2) + (C+x)(C^2 - 2Cx + x^2) + 2Cy^2}{y[C^2 + 2Cx + x^2 - C^2 + 2Cx - x^2]} \\ \frac{\partial p / \partial x}{\partial p / \partial y} &= -\frac{2C^3 - 2Cx^2 + 2Cy^2}{4Cxy} = -\frac{C^2 - x^2 + y^2}{2xy} \end{aligned} \quad (18)$$

Equation 18 tells us the equation for the gradient ratio, Eq. 13.

Next, from geometric considerations we can write the equation of the circle through the wells and through point x, y . The equation is,

$$(y+B)^2 + x^2 = C^2 + B^2 \quad (19)$$

Note that Eq. 19 has its center at location, $-B$ on the y axis, and the radius squared is $C^2 + B^2$, as it should be.

Equation 19 can be solved for x^2 ,

$$\begin{aligned} x^2 &= C^2 + B^2 - (y + B)^2 \\ &= C^2 - y^2 - 2By \end{aligned} \quad (20)$$

and this definition for x^2 can be substituted into Eq. 18.

$$\begin{aligned} \frac{\partial p / \partial x}{\partial p / \partial y} &= -\frac{C^2 - (C^2 - y^2 - 2By) + y^2}{2xy} \\ &= -\frac{2By + 2y^2}{2xy} = -\frac{B + y}{x} \end{aligned} \quad (21)$$

Note that Eq. 21 is exactly the same as Eq. 14! Thus the flow line must be along the circle at all x, y values defined by that circle. This result has even broader significance. Remember that we did not specify the value of B . Thus we can conclude that any circle that can be drawn through the two wells will define a flow line between them.

One last point should be made about these circular flow lines. The one I drew showed a short arc from the injector to the producer. It should be clear that the circle also can traverse an arc 180 degrees from the one shown, going down and to the right from the injection well. This flow line will eventually reach the producer after covering a large circular path, eventually reaching the producer from below and from the left. All circular flow paths exhibit dual directionality, and some flow paths are extremely long. The flow path that moves horizontally to the right of the injection well follows an infinite circle coming to the producer horizontally from the left. In this case, $B = \infty$, and of course, $\partial p / \partial y = 0$.

To summarize our results, the flow lines on a double are circles defined by Eq. 19. Since they are, it should be relatively easy to trace them analytically. We should be able to calculate the exact location of the front as it radiates away from the injection well. Using this idea, we should be able to calculate the time to breakthrough of any flow line and thus calculate the production history of the injected fluids using line integrals along these known flow lines.

2. AREAL SWEEP BEHAVIOR IN A DOUBLET SYSTEM

Since we have exactly defined the equations for the constant pressure lines and the flow lines in a steady state doublet system (injection rate = production rate), it should be possible to use this knowledge to predict the flooding behavior of the system when the mobility ratio is unity. Actually, the concepts involved will not be limited to the doublet problem alone, but I'll start with this because it is simpler and thus the ideas are easier to visualize and compute.

The ideas involved in predicting displacement behavior are relatively simple. They are based on combining Darcy's Law with a material balance to keep track of the front locations. Darcy's Law in vector notation in the x direction is

$$\bar{u}_x = \frac{q_x}{A} = -\frac{k_x}{\mu} \left(\frac{\partial \bar{p}}{\partial x} \right) \quad (1)$$

where \bar{u}_x = Darcy velocity in the x -direction

The true velocity of an injected fluid, \bar{v}_x is greater than the Darcy velocity by a factor depending on the porosity of the system, ϕ , and the change in saturation that occurs when one fluid is displacing another, ΔS .

$$\bar{v}_x = \frac{\bar{u}_x}{\phi \Delta S} = -\frac{k_x}{\mu \phi \Delta S} \left(\frac{\partial \bar{p}}{\partial x} \right) \quad (2)$$

In general, we can define the velocity as a time derivative of distance.

$$v_x = \frac{dx}{dt} \quad (3a)$$

or, upon rearranging,

$$t = \int dt = \int \frac{dx}{v_x} \quad (3b)$$

We can substitute Eq. 2 into Eq. 3b to get,

$$t = -\frac{\mu \phi \Delta S}{k_x} \int \frac{dx}{(\partial p / \partial x)} \quad (4a)$$

Equation 4a is a line integral that defines the time required for a fluid particle to move a distance x under the influence of a known pressure field, $(\partial p / \partial x)$, which is a function of x . I've shown this line integral in the x direction only, however, it can properly be shown in any general direction, s , or in the y direction, as follows,

$$t = -\frac{\mu\phi\Delta S}{k_s} \int \frac{ds}{(\partial p / \partial s)} = -\frac{\mu\phi\Delta S}{k_x} \int \frac{dx}{(\partial p / \partial x)} = -\frac{\mu\phi\Delta S}{k_y} \int \frac{dy}{(\partial p / \partial y)} \quad (4b)$$

I will continue to show the equations in the x direction for convenience in this development, but we should not forget that the coordinate systems are interchangeable, for we will use the idea later to calculate numbers.

2.1 AREAL SWEEP AT BREAKTHROUGH

In general, the area covered by the injected fluid as a function of time can be calculated by a simple material balance, assuming that no injected fluid has yet been produced,

$$A = \frac{qt}{h\phi\Delta S} \quad (5)$$

and when Eq. 4b is substituted into Eq. 5, the result is,

$$A = -\frac{q\mu\phi\Delta S}{hk_x\phi\Delta S} \int \frac{dx}{(\partial p / \partial x)} \quad (6)$$

which relates the total area covered by the injected fluid to the distance traveled for any streamline of interest. In particular, we can see that if the breakthrough streamline is integrated from the injection well to the producing well, it will be possible, from Eq. 6, to calculate the areal sweep at breakthrough. Note that the $\phi\Delta S$ terms cancel in Eq. 6 as we should have expected.

Next we need to look at the pressure field for the doublet system. With the injection well on the positive x axis at $x = C$. This equation was shown in Section 1, Eq. 15, p. 9.

$$\frac{4\pi kh}{q\mu} [p(x, y) - p_i] = -\ln[(C - x)^2 + y^2] + \ln[(C + x)^2 + y^2] \quad (7)$$

Equation 7 can be differentiated with respect to x , to get,

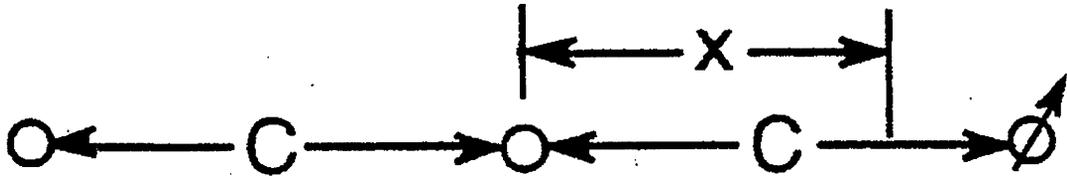
$$\frac{4\pi kh}{q\mu} \frac{\partial p(x,y)}{\partial x} = \frac{2(C-x)}{(C-x)^2 + y^2} + \frac{2(C+x)}{(C+x)^2 + y^2} = f'(x,y) \quad (8)$$

In Eq. 8, I've shown the actual derivative, but have also shown it symbolically as $f'(x,y)$. Equation 8 can be substituted into Eq. 6, and if we assume that k is isotropic as we did in Eq. 7, the result is,

$$A = -4\pi \int_0^x \frac{dx}{f'(x,y)} \quad (9)$$

I've shown Eq. 9 as a line integral in x , but remember it can be defined along any convenient coordinate as shown in Eq. 4b, p. 12. Note also that to tie Eq. 8 to Eq. 6, it is necessary to assume isotropic permeabilities, since Eq. 8 makes that assumption. We should remember, however, that a system with simple anisotropy can be changed into a equivalent isotropic system by coordinate transformation, as was discussed in my notes on injectivity (Brigham, 1985). So this limitation is not a serious practical problem for defining the flow behavior.

The doublet system is shown on the sketch below.



It should be clear from this sketch that the pressure field will be completely symmetric, with pressures greater than p_i to the right of the origin, and pressures lower than p_i to the left. Because of this behavior, the time necessary for a particle of fluid to move to the center, at $x=0$, will be exactly half the time necessary for breakthrough to the production well. Also, it should be clear that the breakthrough streamline will be along the horizontal line directly between the two wells, at $y=0$. With the knowledge, Eq. 8 simplifies to,

$$f'(x,y)_{BT} = \frac{2}{C-x} + \frac{2}{C+x} = \frac{4C}{C^2 - x^2} \quad (10)$$

Notice that Eq. 10 defines x from the origin while the line integral of Eq. 9 defines x from the injection well. This problem can be handled simply by reversing the definition of dx and by changing the limits on the integral of Eq. 9. When these changes are made and when Eq. 10 is substituted in Eq. 9 the result is,

$$\begin{aligned}
 A &= -\frac{4\pi}{4C} \int_C^0 (C^2 - x^2) dx \\
 &= -\frac{\pi}{C} \left[C^2 x - \frac{x^3}{3} \right]_C^0 = +\frac{\pi}{C} \left(C^3 - \frac{C^3}{3} \right) \\
 &= \frac{2\pi C^2}{3}
 \end{aligned} \tag{11}$$

The area swept at breakthrough to the producing well is obviously twice this value. Thus the area swept at breakthrough, A_{BT} , is

$$A_{BT} = \frac{4\pi C^2}{3} \tag{12}$$

where

A_{BT} = Area swept at breakthrough

C = Distance from a well to the center

2.2 BREAKTHROUGH HISTORY

Equation 12 defines the area swept for the breakthrough streamline, which is useful, but we would like to compare this with the behavior of the other streamlines involved in the flooding pattern. To do this, we need to define the general line integrals for all the streamlines. This can be done through our knowledge of the equations for the streamlines and through the use of line integrals. Let us repeat the general picture of the flooding pattern, that we used earlier in Section 1, p. 8, defining the geometry used. This is shown on the next page. Any streamline is a circle, as shown on the sketch, and as defined in Section 1. Its center is on the y axis at location, $-B$. Since it is a circle, its radius, R , is defined as follows,

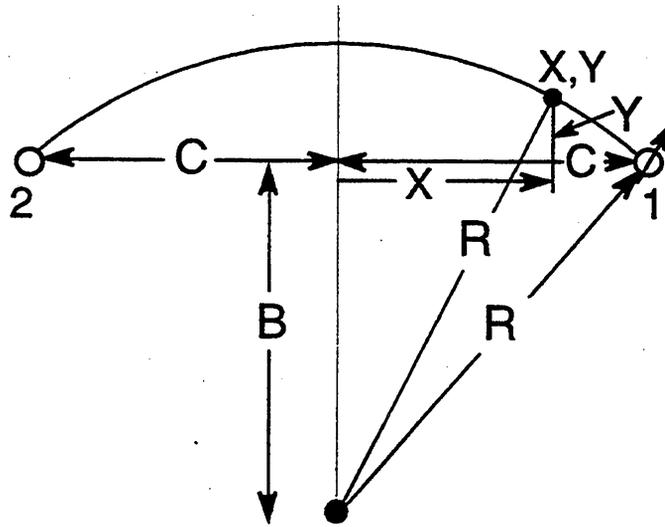
$$R^2 = B^2 + C^2 \tag{13}$$

and the equation of the circle is,

$$(y+B)^2 + x^2 = C^2 + B^2 = R^2 \quad (14a)$$

or

$$x^2 = C^2 - 2By - y^2 \quad (14b)$$



In Section 1, p. 9, we defined the general equation for the pressure gradient in this system, as a function of y and x . The denominators of these pressure gradients were the same whether we were defining $\partial p / \partial x$ or $\partial p / \partial y$. They were,

$$\text{Denominator} = [(C-x)^2 + y^2][(C+x)^2 + y^2] \quad (15a)$$

which simplifies to,

$$\text{Denominator} = (C^2 - x^2)^2 + y^2(2C^2 + 2x^2 + y^2) \quad (15b)$$

To perform the line integral, we will put this expression in terms of y only, using Eq. 14b to substitute for x^2 in Eq. 15b. When these substitutions are made, after considerable algebra, the equation simplifies to,

$$\text{Denominator} = 4y^2(B^2 + C^2) = 4R^2y^2 \quad (15c)$$

The numerator for $\partial p / \partial y$ was defined in Section 1, Eq. 17, p. 9, to be,

$$\begin{aligned} \text{Numerator for } \partial p / \partial y &= 2y(C-x)^2 - 2y(C+x)^2 \\ &= -8Cxy \end{aligned} \quad (16a)$$

and when Eq. 14b is substituted, it becomes

$$\text{Numerator for } \partial p / \partial y = -8Cy\sqrt{C^2 - 2By - y^2} \quad (16b)$$

To calculate the breakthrough behavior of any streamline we can integrate from the injection well to the center, and double it, since the left and right sides are mirror images. The line integral we need to evaluate, in y coordinates, comes from Eqs. 15c and 16b. It is,

$$-\int_0^y \frac{dy}{(\partial p / \partial y)} = \int_0^y \frac{4R^2 y^2 dy}{8Cy\sqrt{C^2 - 2By - y^2}} \quad (17a)$$

or, since $R^2 = C^2 + B^2$, the equation simplifies to,

$$\int_0^y \frac{dy}{(\partial p / \partial y)} = \int_0^y \frac{R^2 y dy}{2C\sqrt{R^2 - (B+y)^2}} \quad (17b)$$

Equation 17b can be expressed in more convenient variables for easier integration. Let's define,

$$z = B + y \quad dy = dz$$

and when,

$$y=0 \quad z = B$$

When these variable changes are made, Eq. 17b becomes,

$$\begin{aligned} -\int_0^y \frac{dy}{(\partial p / \partial y)} &= \frac{1}{2C} \int_B^z \frac{R^2 (z-B) dz}{\sqrt{R^2 - z^2}} \\ &= \frac{1}{2C} \int_B^z \frac{R^2 z dz}{\sqrt{R^2 - z^2}} - \frac{1}{2C} \int_B^z \frac{R^2 B dz}{\sqrt{R^2 - z^2}} \end{aligned} \quad (18)$$

The integrals shown in Eq. 18 are common standard forms. When evaluated and combined they become,

$$\begin{aligned} -\int_0^y \frac{dy}{(\partial p / \partial y)} &= -\frac{R^2}{2C} \left[(R^2 - z^2)^{1/2} + B \sin^{-1} \left(\frac{z}{R} \right) \right]_B \\ &= -\frac{R^2}{2C} \left[(R^2 - z^2)^{1/2} - C + B \left[\sin^{-1} \left(\frac{z}{R} \right) - \sin^{-1} \left(\frac{B}{R} \right) \right] \right] \end{aligned} \quad (19)$$

In general, we are performing this integral to the time when the fluid has reached the center of the system at $x = 0$. At this time, $y + B = R$, thus $z = R$, and the first term in the brackets of Eq. 19 disappears. Thus the integral of Eq. 19 to the midpoint between the wells becomes,

$$-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} = \frac{R^2}{2} \left[1 - \frac{B}{C} \left[\sin^{-1}(1) - \sin^{-1} \left(\frac{B}{R} \right) \right] \right] \quad (20)$$

It would be well to check on whether Eq. 20 properly degenerates to Eq. 11 as the angle of the streamline approaches zero; that is, as the value of B approaches infinity. In general under this condition, we can define B as follows,

$$B = nC \quad (21)$$

where $n = A$ large number, approaching infinity

then, R^2 becomes,

$$R^2 = B^2 + C^2 = n^2 C^2 + C^2 = (n^2 + 1) C^2 \quad (22)$$

Also we can cast the arc sine terms as arc tangents, for there are convenient series expressions available for arc tangents.

$$\sin^{-1} \left(\frac{B}{R} \right) = \tan^{-1} \left(\frac{B}{C} \right) = \tan^{-1}(n) \quad (23a)$$

$$\sin^{-1}(1) = \tan^{-1}(\infty) = \frac{\pi}{2} \quad (23b)$$

Thus Eq. 20, at a small angle where B approaches infinity, becomes

$$-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} = \frac{R^2}{2} \left[1 - \frac{B}{C} \left[\frac{\pi}{2} - \tan^{-1}(n) \right] \right] \quad (24)$$

The infinite series expression for arc tangent n where n is large is,

$$\tan^{-1}(n) = \frac{\pi}{2} - \frac{1}{n} + \frac{1}{3n^3} - \frac{1}{5n^5} + \dots \quad (25)$$

Substituting Eq. 25 into Eq. 24 results in,

$$-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} = \frac{R^2}{2} \left[1 - \frac{B}{C} \left[\frac{1}{n} \left(1 - \frac{1}{3n^2} + \frac{1}{5n^4} - \dots \right) \right] \right] \quad (26a)$$

and when the definitions for R and B from Eqs. 21 and 22 are substituted into Eq. 26a, it becomes,

$$-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} = \frac{(n^2 + 1)C^2}{2} \left[1 - \frac{nC}{C} \left[\frac{1}{n} \left(1 - \frac{1}{3n^2} + \frac{1}{5n^4} - \dots \right) \right] \right] \quad (26b)$$

$$= \frac{C^2}{6} \left(1 + \frac{1}{n^2} \right) \left(1 - \frac{3}{5n^2} + \dots \right) \approx \frac{C^2}{6} \left(1 + \frac{2}{5n^2} \right) \quad (26c)$$

Equation 26c clearly shows that the line integral approaches the value of $C^2/6$ as the value of n approaches infinity.

Equation 9, p. 14, shows the area swept at this time as a function of a line integral in x . However, as indicated in Eqs.

4b, p. 12, this line integral can be cast in any convenient coordinates. The variable used in Eq. 26c was y . Thus the

value for area swept at the time the injected fluid reaches the center of the system is,

$$A = 4\pi \left(\frac{C^2}{6} \right) = \frac{2\pi C^2}{3} \quad (27a)$$

and the area swept at the time the injected fluid reaches the production well is exactly double this value,

$$A_{BT} = 2 \left(\frac{2\pi C^2}{3} \right) = \frac{4\pi C^2}{3} = 1.3333\pi C^2 \quad (27b)$$

Note that Eq. 27b gives exactly the same result as Eq. 12, p. 14, did. Thus we have shown that Eq. 19 is a valid expression for the behavior of the flow in a general streamline.

2.2.1 Area Swept at Times When Various Streamline Break Through

In general, we've shown that the following line integral, Eq. 20 is proportional to the injection volume necessary for the injected fluid to reach the center of the pattern,

$$-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} = \frac{R^2}{2} \left[1 - \frac{B}{C} \left[\sin^{-1}(1) - \sin^{-1}\left(\frac{B}{R}\right) \right] \right] \quad (20)$$

To break through to the producing well, this value must be doubled. In addition, to change the integral to account for the size of the system, from Eq. 9, p. 13, we must multiply by 4π . So the equation relating the area of fluid injected to the breakthrough streamline of interest is,

$$A_{\alpha} = 4\pi R^2 \left[1 - \frac{B}{C} \left[\frac{\pi}{2} - \sin^{-1} \left(\frac{B}{R} \right) \right] \right] \quad (28)$$

where A_{α} equals the area of fluid injected (and is proportional to the volume injected since ϕ , k and ΔS are assumed constant) at the time the streamline, defined by angle α breaks through.

We will look at Eq. 28 at various breakthrough angles to calculate the production history of this pattern, but first we need to define the fractional flow relationships. To this end, it should be clear that the fraction of injected fluid that is being produced is exactly proportional to the angle that has just broken through. Let me illustrate by an example. Suppose that the 30° streamline has just broken through to the producing well. Clearly all the fluid being produced from 0 to 30° is now injected fluid, while all the remaining produced fluid is original in-place fluid up to an angle of 180° . Since the breakthrough streamlines are symmetric about the x axis, this same statement holds true for the other half of the producing well. So in this case we can state that the fractional flow of injection fluid being produced is,

$$f = \frac{30}{180} = \frac{60}{360} = 0.1667 \quad (29)$$

This same general concept will hold for any breakthrough streamline. With this thought in mind, we are now in a position to calculate the production history of the injected fluid.

2.2.2 Breakthrough Angle 30°

When the 30° streamline breaks through the fractional flow will be 0.1667, as calculated in Eq. 29. At this angle, the values for the geometric parameters of Eq. 28 are,

$$\begin{aligned} R &= 2C \\ B &= \sqrt{3} C \\ \sin^{-1} \left(\frac{B}{R} \right) &= \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = 60^{\circ} = \frac{\pi}{3} \end{aligned}$$

and substituting into Eq. 28 produces,

$$\begin{aligned}
 A_{30^\circ} &= 4\pi(4C^2) \left[1 - \frac{\sqrt{3}C}{C} \left(\frac{\pi}{2} - \frac{\pi}{3} \right) \right] \\
 &= 16\pi C^2 \left(1 - \frac{\sqrt{3}\pi}{6} \right) = 16\pi C^2 (0.09310) \\
 &= 1.4896\pi C^2
 \end{aligned} \tag{30}$$

Note that the constant in Eq. 30, 1.4896, is only slightly greater than the breakthrough constant, 1.3333, from Eq. 27b. In brief, this result tells us that the fractional flow of produced fluid increases rapidly shortly after breakthrough. This behavior cannot continue for the entire flood life, for it should be clear from the pattern geometry that an infinite amount of fluid must be injected before the produced fractional flow reaches 1.0.

2.2.3 Breakthrough Angle 60°

At this breakthrough angle, the fractional flow is,

$$f = \frac{60}{180} = \frac{120}{360} = 0.3333 \tag{31}$$

and the appropriate parameter definitions are,

$$\begin{aligned}
 B &= \frac{C}{\sqrt{3}} = \frac{\sqrt{3}C}{3} \\
 R &= 2B = \frac{2\sqrt{3}C}{3} \\
 \sin^{-1} \left(\frac{B}{R} \right) &= \sin^{-1} \left(\frac{1}{2} \right) = 30^\circ = \frac{\pi}{6}
 \end{aligned}$$

thus, Eq. 28 becomes

$$\begin{aligned}
 A_{60^\circ} &= 4\pi \left(\frac{12C^2}{9} \right) \left[1 - \frac{\sqrt{3}}{3} \left(\frac{\pi}{2} - \frac{\pi}{6} \right) \right] \\
 &= \frac{48\pi C^2}{9} \left(1 - \frac{\sqrt{3}\pi}{9} \right) \\
 &= \frac{48\pi C^2}{9} (0.29540) = 2.1088\pi C^2
 \end{aligned} \tag{32}$$

Again, the areal volume injected to achieve $f = 0.3333$ is relatively modest. Only about 1.58 times the breakthrough volume has increased the fractional flow to this level.

2.2.4 Breakthrough Angle 90°

At this breakthrough angle, the fractional flow is

$$f = \frac{90}{180} = \frac{180}{360} = 0.5000 \quad (33)$$

This breakthrough angle is interesting, for it is the case where exactly half the produced fluid is original fluid in place, and half is injected fluid. Further, the breakthrough streamline lies on a circle centered on the origin of the coordinate system, as indicated by the geometric parameters.

$$\begin{aligned} B &= 0 \\ R &= C \\ \sin^{-1}\left(\frac{B}{R}\right) &= \sin^{-1}(0) = 0^\circ \end{aligned}$$

and Eq. 28 becomes,

$$A_{90^\circ} = 4\pi C^2(1-0) = 4\pi C^2 \quad (34)$$

This volume is three times the breakthrough value, thus the fractional flow is not rising as rapidly as it was at first, as we anticipated. Because of this behavior, subsequent calculations will be made using smaller intervals of fractional flow and angle.

2.2.5 Breakthrough Angle 105°

At this breakthrough angle, the fractional flow is,

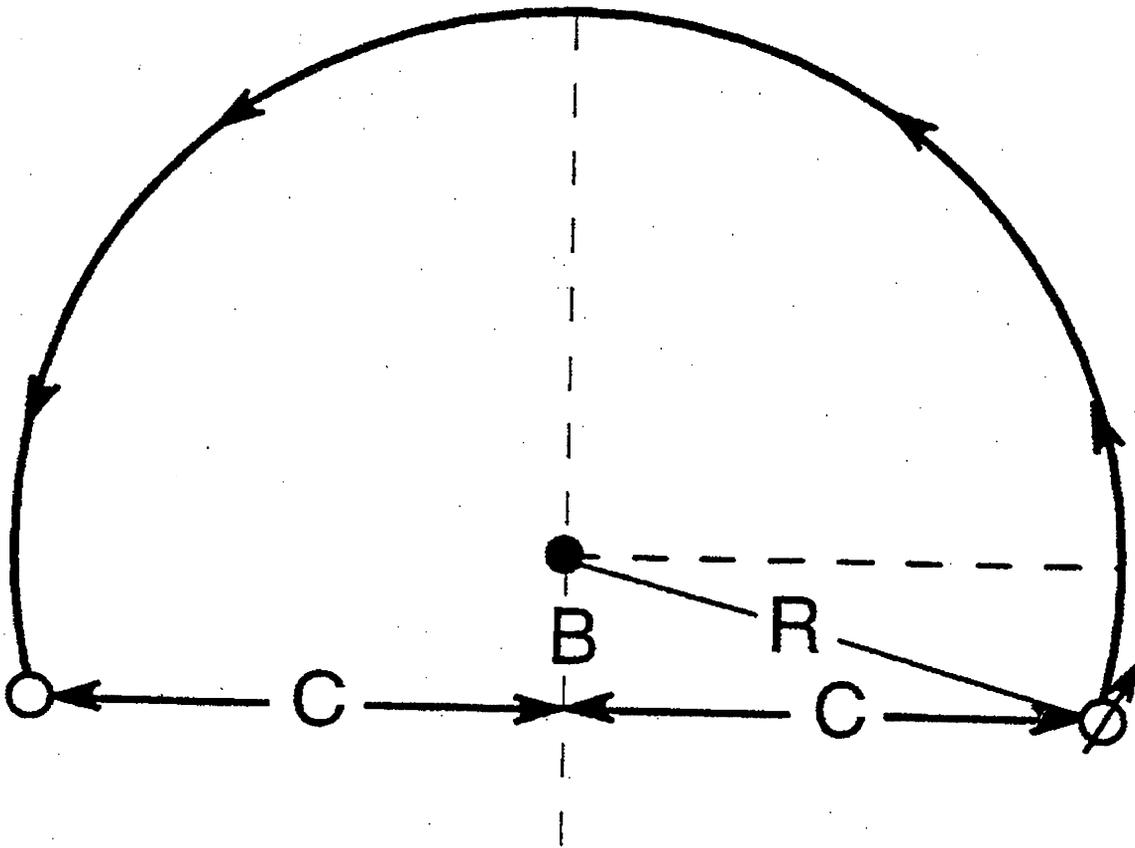
$$f = \frac{105}{180} = \frac{210}{360} = 0.5833 \quad (35)$$

This is an odd angle, so the relationships between the geometric parameters will not be as simple as before. A sketch of this system might help define the geometric parameters as shown on the next page. Note that this streamline subtends more than half a circle. With an angle of 105°, the right triangle drawn in the sketch has an angle to the well of 15°

(105-90=15). Also the value of B is now negative as we have defined it. Thus the parameters can be defined as follows,

$$\tan(15^\circ) = 0.26795 = -\frac{B}{C}$$

$$B = -0.26795C$$



and $\cos(15^\circ) = 0.965932 = \frac{C}{R}$

or $R = 1.03528C$

Using these parameters in the breakthrough equation, Eq. 28, we get,

$$A_{105^\circ} = 4\pi(1.03528C)^2 \left[1 + 0.26795 \left(\frac{\pi}{2} + \frac{15(2\pi)}{360} \right) \right]$$

$$= 4.28722\pi C^2 [1 + 0.26795(0.58333\pi)]$$

$$= 6.3924C^2$$
(36)

Notice, in Eq. 36, that since B/C is now negative, the bracket term at the right is now added to 1.00 rather than subtracted, as it was when Eq. 28 was used for angles less than 90°. Also notice that the term, $\sin^{-1}(B/R)$, is also added to $\pi/2$, rather than subtracted, for the same reason (the angle is greater than 90°). This combination of factors causes the volume injected term to increase rapidly, as we might have suspected upon viewing the large circular segment the injected fluid must traverse.

2.2.6 Breakthrough Angle 120°

At this angle, the fractional flow is,

$$f = \frac{120}{180} = \frac{240}{360} = 0.6667 \quad (37)$$

and the parameters in the breakthrough equation are,

$$\begin{aligned} -B &= \frac{C}{\sqrt{3}} = \frac{\sqrt{3}C}{3} \\ R &= -2B = \frac{2\sqrt{3}C}{3} \\ R^2 &= \frac{4C^2}{3} \\ \sin^{-1}\left(\frac{B}{R}\right) &= \sin^{-1}\left(-\frac{1}{2}\right) = -30^\circ = -\frac{\pi}{6} \end{aligned}$$

When these parameters are inserted into Eq. 28, it becomes,

$$\begin{aligned} A_{120^\circ} &= 4\pi \left(\frac{4C^2}{3} \right) \left[1 + \frac{\sqrt{3}}{3} \left(\frac{\pi}{2} + \frac{\pi}{6} \right) \right] \\ &= 11.7824\pi C^2 \end{aligned} \quad (38)$$

Notice the volume is increasing rapidly, as expected.

2.2.7 Breakthrough Angle 135°

At this angle, the fractional flow becomes,

$$f = \frac{135}{180} = \frac{270}{360} = 0.7500 \quad (39)$$

The appropriate geometric terms for Eq. 28 are,

$$\begin{aligned}
 R &= \sqrt{2}C \\
 R^2 &= 2C^2 \\
 B &= -C \\
 \sin^{-1}\left(\frac{B}{R}\right) &= \sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) = -45^\circ = -\frac{\pi}{4}
 \end{aligned}$$

and the breakthrough equation becomes,

$$\begin{aligned}
 A_{135^\circ} &= 4\pi(2C^2)\left[1 + 1\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right] \\
 &= 26.8496\pi C^2
 \end{aligned} \tag{40}$$

2.2.8 Breakthrough Angle 150°

At an angle of 150°, the fractional flow is,

$$f = \frac{150}{180} = \frac{300}{360} = 0.8333 \tag{41}$$

The geometric parameters are,

$$\begin{aligned}
 R &= 2C \\
 R^2 &= 4C^2 \\
 B &= -\sqrt{3}C \\
 \sin^{-1}\left(\frac{B}{R}\right) &= \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -60^\circ = -\frac{\pi}{3}
 \end{aligned}$$

and the breakthrough equation becomes,

$$\begin{aligned}
 A_{150^\circ} &= 4\pi(4C^2)\left[1 + \sqrt{3}\left(\frac{\pi}{2} + \frac{\pi}{3}\right)\right] \\
 &= 88.5520\pi C^2
 \end{aligned} \tag{42}$$

No further angles will be calculated, for the volume terms have already become excessive for any practical purposes.

There were many calculations and results presented here. It would probably be best to summarize them into a table for close perusal, and also to graph the results to see the breakthrough behavior. The following table summarizes those results, and the figure on the following page shows them graphically.

Breakthrough Behavior of the Doublet System

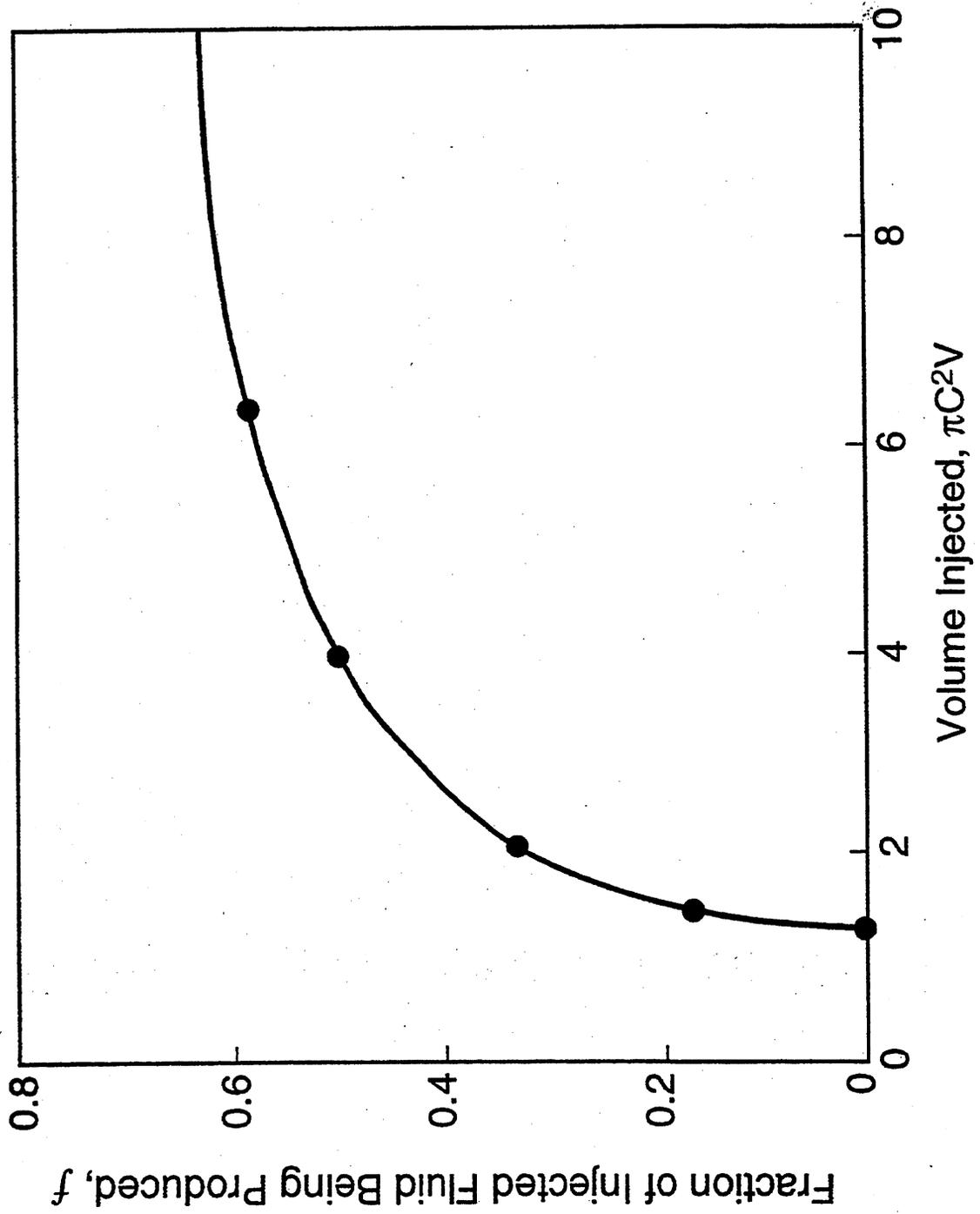
Angle Degrees	Fractional Flows, f	R/C	B/C	Injected Volume Constant
0	0	∞	∞	1.3333
30	0.1667	2.0	$\sqrt{3}$	1.4896
60	0.3333	$2\sqrt{3}/3$	$\sqrt{3}/3$	2.1088
90	0.5000	1.0	0	4.0000
105	0.5833	1.03528	-0.26795	6.3924
120	0.6667	$2\sqrt{3}/3$	$-\sqrt{3}/3$	11.7824
135	0.7500	$\sqrt{2}$	-1.0	26.8496
150	0.8333	2.0	$-\sqrt{3}$	88.5520

These results clearly show that the fraction of injected fluid being produced rises quite rapidly immediately after breakthrough, up to a fractional flow of about 30% to 50%. Thereafter, the fractional flow rises at a rapidly declining rate such that more than 20 breakthrough pore volumes are required to increase the fractional flow to 75%, and of course, $f = 1.0$ requires an infinite injection volume.

2.3 FLOODFRONT LOCATIONS

We've developed equations for a doublet, which define the volume that must be injected for breakthrough of any designated streamline. Another useful exercise would be to define the locations of the flood fronts of the streamlines at any time (that is, at any given injection volume). This is a general problem that could be solved at any time, but the illustrative purposes let us look at flood front locations for other streamlines at a time when the first streamline breaks through.

Breakthrough Behavior of a
Doublet System (Spacing - 2C)



To perform this task we realize that, when the first streamline has broken through to the production well, some of the streamlines will have gone past the center of the system at $x = 0$. These will be the streamlines at small angles. Meanwhile those at large angles will not yet have reached the center of the pattern. The angle at which this difference in behavior occurs will need to be defined, for the equations defining their motion will differ. So first we will define the limiting angle.

In general, the line integral to the center of the pattern is Eq. 20, from p. 17.

$$-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} = \frac{R^2}{2} \left\{ 1 - \frac{B}{C} \left[\sin^{-1}(1) - \sin^{-1}\left(\frac{B}{R}\right) \right] \right\} \quad (20)$$

and the line integral for first breakthrough, from Eq. 9 or Eq. 27b, is,

$$-4\pi \int \frac{dy}{\partial p / \partial x} = (4\pi) \left(\frac{C^2}{3} \right) \quad (12) \text{ or } (27b)$$

To define the streamline that has just reached the center, we set the two line integrals equal to each other, eliminating the 4π term from Eq. 27b, since it is not included in Eq. 20.

$$\frac{C^2}{3} = \frac{R^2}{2} \left\{ 1 - \frac{B}{C} \left[\sin^{-1}(1) - \sin^{-1}\left(\frac{B}{R}\right) \right] \right\} \quad (43a)$$

or,

$$1 - \frac{2}{3} \left(\frac{C}{R} \right)^2 = \frac{B}{C} \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{B}{R}\right) \right] \quad (43b)$$

There are three ratios shown in Eq. 43b, C/R , B/C and B/R . It would be convenient if we could put all three in terms of a single ratio. This can be done easily if we recognize some simple geometric relationships,

$$R^2 = B^2 + C^2 \quad (13)$$

and,

$$\sin^{-1}\left(\frac{B}{R}\right) = \tan^{-1}\left(\frac{B}{C}\right) \quad (23a)$$

Now Eq. 43b can be written in terms of B/C only, after a bit of simple algebra,

$$\frac{B^2}{C^2} + \frac{1}{3} = \left(\frac{B}{C} \right) \left(\frac{B^2}{C^2} + 1 \right) \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{B}{C}\right) \right] \quad (43c)$$

Now Eq. 43c can be solved for B/C . Since it contains the arc tangent term, the solution must be made by trial and error. The result is,

$$\frac{B}{C} = 0.306945 \quad (44a)$$

which also results in,

$$\frac{R}{C} = 1.046047 \quad (44b)$$

and, the angle of this limiting streamline is,

$$\text{Angle} = 72.94^\circ = 1.27298 \text{ Radians} \quad (44c)$$

2.3.1 Angles Less Than 72.94°

Streamlines for an angle less than 72.94° go beyond the center of the pattern. Thus to calculate their line integrals it is easier if two integrations are performed. One is from the injection well to the center. The other is the mirror image from the center to the final location. This is done as follows.

The line integral to the center has been defined already. It is Eq. 20, p. 17.

$$-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} = \frac{R^2}{2} \left\{ 1 - \frac{B}{C} \left[\frac{\pi}{2} - \sin^{-1} \left(\frac{B}{R} \right) \right] \right\} \quad (20)$$

In the first section of these notes, this integral has been evaluated to many angles.

To calculate the line integral beyond the center we should recognize that it is the same as if we were integrating from the mirror image point to the center in the right half plane; that is, from the final value of y to the center. Mathematically that is the same as integrating to the center and subtracting the integral from zero to y . The integration to the center is again Eq. 20, and the integration to y is the same as performed before, Eq. 17b, p. 16.

$$-\int_0^y \frac{dy}{\partial p / \partial y} = \int_0^y \frac{R^2 y dy}{2C \sqrt{R^2 - (B+y)^2}} \quad (17b)$$

With the coordinate change, $z = B + y$, this integral has already been evaluated in Eq. 19, p. 17,.

$$-\int_0^y \frac{dy}{(\partial p / \partial y)} = -\frac{R^2}{2C} \left\{ \left(R^2 - z^2 \right)^{1/2} - C + B \left[\sin^{-1} \left(\frac{z}{R} \right) - \sin^{-1} \left(\frac{B}{R} \right) \right] \right\} \quad (19)$$

Thus, we merely subtract Eq. 19 from Eq. 20 to evaluate the line integral beyond the centerline. After canceling like terms, the result is,

$$-\int_y^{R-B} \frac{dy}{(\partial p / \partial y)} = -\frac{R^2}{2C} \left\{ \left(R^2 - z^2 \right)^{1/2} - B \left[\left(\frac{\pi}{2} \right) - \sin^{-1} \left(\frac{z}{R} \right) \right] \right\} \quad (45)$$

In general we've already shown the integral for the breakthrough streamline. For example, from Eqs. 9 and 27b (with the 4π term removed), the integral is,

$$-\int_0^{2C} \frac{dx}{(\partial p / \partial x)} = \frac{C^2}{3} \quad (12) \text{ or } (27b)$$

Thus, for any of the streamlines that go pass the center of the system, we can now write,

$$\int_0^{2C} \frac{dx}{(\partial p / \partial x)_{BT}} = \int_0^{R-B} \frac{dy}{(\partial p / \partial y)_\alpha} + \int_y^{R-B} \frac{dy}{(\partial p / \partial y)_\alpha} \quad (46a)$$

or,

$$\text{Equation 12} = \text{Equation 20} + \text{Equation 45} \quad (46b)$$

Notice that in Eq. 46a I have subscripted the $\partial p / \partial x$ term with BT to indicate the breakthrough integral, and the $\partial p / \partial y$ terms are subscripted with α to indicate they are for the angle of interest. Equations 46a and 46b merely state, with the limits and subscripts shown, that we are looking at these integrals at the same time. At any particular angle, in general, Eqs. 12 and 20 have already been solved, so only Eq. 45 must be solved. But it would be more convenient to write it in terms of y rather than z . With this change in variable, Eq. 45 becomes,

$$-\int_y^{R-B} \frac{dy}{(\partial p / \partial y)} = -\frac{R^2}{2C} \left\{ \left(R^2 - (y+B)^2 \right)^{1/2} - B \left[\left(\frac{\pi}{2} \right) - \sin^{-1} \left(\frac{y+B}{R} \right) \right] \right\} \quad (47)$$

Thus the heart of the remaining calculations will be to evaluate Eq. 47 at various angles, up to an angle of 72.9°.

2.3.2 Streamline Angle 15°

Definition of the flood front shape requires accurate detail near the front where there will be a sharp point. So I will first calculate the 15° streamline. This angle was not addressed before, so in addition to evaluating Eq. 45 at 15°

we will also need to evaluate Eq. 20 at 15°. That will be done next.

$$\begin{aligned}\tan 15^\circ &= \frac{C}{B} = 0.267949 \\ B &= 3.732051C \\ \sin 15^\circ &= \frac{C}{R} = 0.258819 \\ R &= 3.863703C \\ \sin^{-1}\left(\frac{B}{R}\right) &= 75^\circ = \frac{5\pi}{12}\end{aligned}$$

Thus Eq. 20 evaluated at 15° becomes,

$$\begin{aligned}-\int_0^{R-B} \frac{dy}{(\partial p / \partial y)} &= \frac{(3.863703^2 C^2)}{2} \left\{ 1 - 3.732051 \left[\frac{\pi}{2} - \frac{5\pi}{12} \right] \right\} \\ &= 0.171311C^2\end{aligned}\quad (48)$$

So for an angle of 15°, Eq. 46 becomes

$$\frac{C^2}{3} - 0.171311C^2 = \frac{R^2}{2C} \left\{ \left[R^2 - (y+B)^2 \right]^{1/2} - B \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{y+B}{R}\right) \right] \right\} \quad (49)$$

Equation 49 must be solved by trial and error for y/C , remembering that R/C and B/C have been defined already by the geometry of the system. The trial and error result is,

$$\frac{y}{C} = 0.04663 \quad (50a)$$

which, from the geometry of the circular system, means,

$$\frac{x}{C} = 0.80609 \quad (50b)$$

This same trial and error procedure can be used to solve for the locations of the fronts on any streamlines up to 72.9°. I'll not detail the equations and calculations involved, for they are rather tedious if straight forward. I will, however, summarize the results in the table on the following page.

Notice in this table that I've labeled the x/C values with negative signs to remind us that the fronts along these streamlines have moved beyond the center of the system. As defined in Eq. 50b, for example, x/C was positive.

**Calculations of Locations of Flood Fronts At Breakthrough of a
Doublet System; Angles Less than 72.94°**

Angle (degrees)	R/C	B/C	Integral to Center (x/C^2)	y/C	x/C
0	∞	∞	0.166667	0.00000	-1.00000
15	3.863703	3.732051	0.171311	0.04663	-0.80609
30	2.000000	$\sqrt{3}$	0.186201	0.17125	-0.61437
45	$\sqrt{2}$	1.000000	0.214602	0.35009	-0.42102
60	$2/\sqrt{3}$	$\sqrt{3}/3$	0.263600	0.55802	-0.21038

2.3.3 Angles Greater than 72.94°

When the angle away from the injection well is greater than 72.9° the streamline remains at the right half plane of the system, so the equation that needs to be solved is merely the fact that Eq. 19, p. 17, equals Eqs. 12 or 27b (with the term, 4π , removed) as follows,

$$\frac{C^2}{3} = -\frac{R^2}{2C} \left\{ \left[R^2 - (y-B)^2 \right]^{1/2} - C + B \left[\sin^{-1} \left(\frac{y+B}{R} \right) - \sin^{-1} \left(\frac{B}{R} \right) \right] \right\} \quad (51)$$

This equation also must be solved by trail and error for y when the values of B and R are specified for any given angle. Later, I will show the results of these calculations, but first I need to show the line integral for the streamline that moves directly to the right of the injector along the positive x axis away from the producer.

The flow along the horizontal line to the right is defined by a line integral in the x direction just as the breakthrough streamline was defined in Eq. 11, p. 14. This equation was a line integral defining the time to the centerline. The time to breakthrough is double that integral. We merely need to set that time to be the same as for the fluid moving to the right, as follows,

$$-2 \int_C^0 (C^2 - x^2) dx = - \int_C^{C+x} (C^2 - x^2) dx \quad (52a)$$

which, when integrated, becomes,

$$-2 \left| C^2 x - \frac{x^3}{3} \right|_C^0 = - \left| C^2 x - \frac{x^3}{3} \right|_C^{C+x}$$

or

$$\frac{4C^3}{3} = Cx^2 + \frac{x^3}{3} \quad (52b)$$

The solution to Eq. 52b is,

$$\frac{x}{C} = 1.00000 \quad (53)$$

Equation 53 is for the distance beyond the injection well. The distance from the center is $x/C = 2.00000$.

Now we can list the results of the calculations for all angles greater than 72.9°. For all but the 180° angle, the calculations were made by trial and error using Eq. 51. To make this presentation more complete, I'll list the results for all angles calculated from 0 to 180°.

Notice, from this table on the following page, that there is a symmetry in the values of R/C and B/C around the 90° angle as we might have expected. There are positive values for B/C below 90° and negative values above 90° due to the way we defined B . Notice that the greatest movement in the y direction is along a streamline that lies between 105° and 120°. This is an interesting result that one probably wouldn't have anticipated without making the calculations.

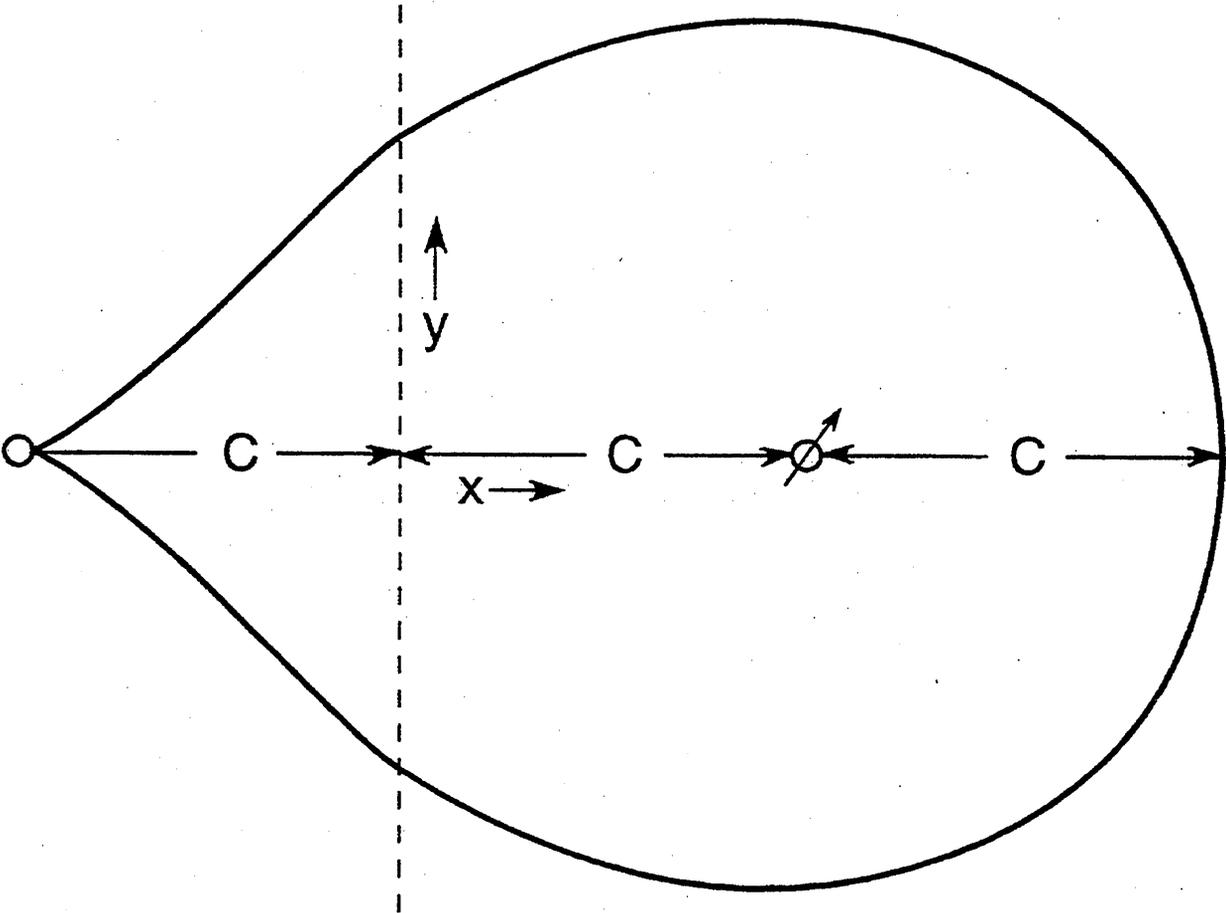
These results are graphed in the attached figure, p. 34. This figure looks a great deal like a circle with tapering wedge attached to it at the left. Although I've not done the calculations here, it would also be interesting and instructive to look at the shapes of swept areas after breakthrough. This calculation would be straight forward to accomplish. It would merely require solving equations similar to Eqs. 46a and 51 for later times that are defined by the breakthrough streamline of interest.

**Calculations for Locations of Flood Fronts At Breakthrough of a
Doublet System; Angles From 0 to 180°**

Angle (degrees)	R/C	B/C	Integral to Center (x/C^2)	y/C	x/C
0	∞	∞	0.166667	0.00000	-1.00000
15	3.863703	3.732051	0.171311	0.04663	-0.80609
30	2.000000	$\sqrt{3}$	0.186201	0.17125	-0.61437
45	$\sqrt{2}$	1.000000	0.214602	0.35009	-0.42102
60	$2/\sqrt{3}$	$\sqrt{3}/3$	0.263600	0.55802	-0.21038
75	1.03528	0.26795		0.76667	0.03691
90	1.000000	0		0.94281	0.33333
105	1.03528	-0.26795		1.05051	0.67779
120	$2/\sqrt{3}$	$-\sqrt{3}/3$		0.94497	1.41314
135	$\sqrt{2}$	-1.00000		0.94497	1.41314
150	2.000000	$-\sqrt{3}$		0.71250	1.72061
165	3.863703	-3.732051		0.38328	1.93870
180	∞	∞		0.00	2.00000

Breakthrough Sweep for a Doublet

$$\text{Area} = 4\pi C^2/3$$



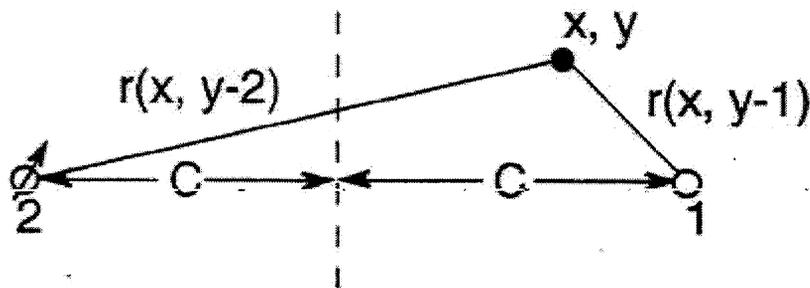
3. PROBLEMS IN THE DOUBLET HALF PLANE

We've looked at the doublet system in some detail to see the nature of the flow lines and constant pressure lines. We found they were circles whose equations could be defined analytically. This knowledge can be used to solve many interesting and useful flow problems related to circles and constant pressure boundaries. In these notes, I will concentrate on problems of interest in the half-plane. Both fluid flow and heat flow problems of interest to a petroleum engineer are addressed.

These problems start with a very simple case which only involves a well and a constant pressure surface. The geometries are gradually increased in complexity with a resulting increase in the complexity of the equations to be solved. The results are compared to forms commonly seen in the literature, and when there are differences, these differences are explained.

3.1. WELL NEAR A CONSTANT PRESSURE BOUNDARY

The simplest of these systems that we will look at, is for a well producing from a semi-infinite system when there is a linear constant pressure boundary near the well. This problem is exactly the same as the general injection production doublet problem discussed earlier, for that problem automatically sets up a linear constant pressure boundary halfway between the wells. A sketch of the system is repeated below.



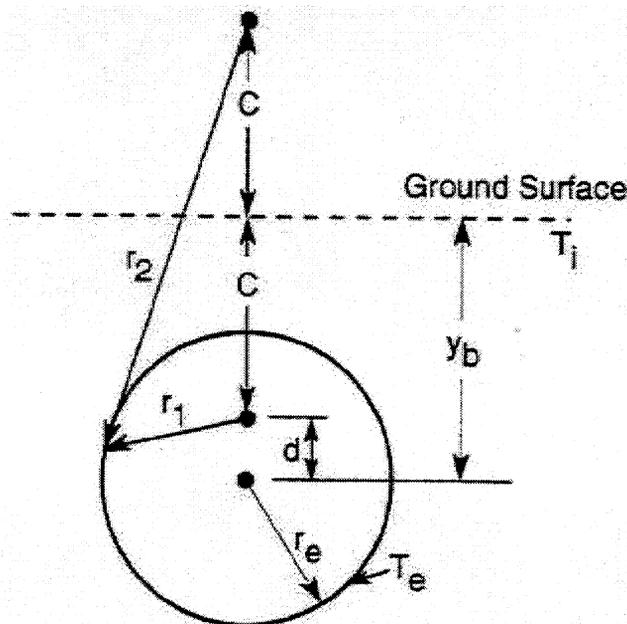
This equation is particularly useful to handle flow in a system where there is natural water influx, or where there is a line drive water injection pattern.

Sometimes in groundwater systems, the aquifers connect directly with the sea bed or lake. The aquifers near San Jose, California, are a case in point, where they connect with San Francisco Bay. In cases like this, Eq. 2b is an exact representation of the pressure and flow field of the systems.

It is also clear that the breakthrough history of such systems can be calculated using ideas similar to those in the previous section. The starting point will be the constant pressure boundary at the bottom of this figure.

3.2. LARGE WELL RADIUS NEAR A CONSTANT PRESSURE BOUNDARY

Obviously, almost all real wells will have quite a small radius compared to the normal spacing seen in oil and gas fields. So the large well problem, at first glance, seems unimportant. But instead, let's consider a buried pipeline near a constant temperature surface and consider the heat loss from that pipeline. Since the heat flow equation is identical in form to the diffusivity equation we use for fluid flow, we should be able to solve this problem exactly from our knowledge of doublet systems. First, let us show a sketch of the system using nomenclature similar to the nomenclature we used before,



The heat flow equivalent to the steady state fluid flow equation for this system is,

$$\frac{2\pi k_h L}{q_h} (T_e - T_i) = \frac{1}{2} \ln \left(\frac{r_2}{r_1} \right)^2 = -\frac{1}{2} \ln K = \ln \sqrt{1/K} \quad (3)$$

In Eq. 3, I've inverted the log term compared to Eq. 1 since T_e is greater than T_i , and the imaginary well at C is an injector. Further I've changed the distance variables to y rather than x , for the symmetry direction is reversed.

The terms in the figure and the equation are,

k_h = Thermal conductivity of the earth

q_h = Steady state rate of heat flow from the constant temperature circle to the constant temperature earth surface

L = Length of pipeline

r_1, r_2 = Radii from the imaginary wells to the constant temperature circle

T_e = Temperature of the constant temperature circle (the pipeline)

T_i = Initial earth temperature— surface temperature

y_b = Average depth of burial of the pipeline

C = Spacing of doublet point source and point sink

d = Distance from point source to center of constant temperature circle, $y_b - C$

K = Constant equal to the ratio r_1^2 / r_2^2 , defined along the constant temperature circle. K ranges from 0 to 1, depending on the depth of burial

In earlier notes I derived the equation for the constant pressure circle. It was written for wells that were along the x axis with zero y . A similar equation can be written for the vertically symmetric system sketched above. It is,

$$\left[y - \frac{C(1+K)}{(1-K)} \right]^2 + x^2 = \frac{4C^2 K}{(1-K)^2} \quad (4)$$

From Eq. 4, the location of the center of the circle, y_b , is defined as follows:

$$y_b = C(1 + K)/(1 - K) \quad (5a)$$

and the radius of the pipeline, r_e , is defined in Eq. 4 as,

$$r_e^2 = 4C^2K/(1 - K)^2$$

or

$$r_e = 2C\sqrt{K}/(1 - K) \quad (6)$$

If we can find a way to relate K to r_e and y_b , then Eq. 3 can be put into useful terms to calculate the heat loss rate from the pipeline. This can be done by relating d to the variables. From Eq. 5a,

$$y_b = C + d = C(1 + K)/(1 - K) \quad (5b)$$

and upon rearranging Eq. 5b, we get,

$$d = \frac{C(1 + K)}{(1 - K)} - C = \frac{2KC}{(1 - K)} \quad (7)$$

Now we can divide Eq. 6 by Eq. 7 to get,

$$\frac{r_e}{d} = \frac{2C\sqrt{K}(1 - K)}{(1 - K) 2KC} = 1/\sqrt{K} \quad (8a)$$

or,

$$\frac{d^2}{r_e^2} = K \quad (8b)$$

which relates K to r_e and d in the figure.

Equation 8 relates d and r_e to K . But since, from the geometry of the system, we can relate d to y_b and C , we can write

$$\frac{d}{r_e} = \frac{y_b}{r_e} - \frac{C}{r_e} = \sqrt{K} \quad (9)$$

The ratio C/r_e is known from Eq. 6, and when this definition is substituted into Eq. 9, we get,

$$\frac{y_b}{r_e} - \frac{(1-K)}{2\sqrt{K}} = \sqrt{K} \quad (10a)$$

which when rearranged becomes,

$$\frac{1}{K} - \frac{2y_b}{r_e\sqrt{K}} + 1 = 0 \quad (10b)$$

Equation 10b is a quadratic in $\sqrt{1/K}$, which from Eq. 3, is exactly the term we need to solve for. Its solution is

$$\frac{1}{\sqrt{K}} = \frac{y_b}{r_e} + \sqrt{(y_b/r_e)^2 - 1} \quad (11)$$

and when Eq. 11 is substituted into Eq. 3 and the equation is rearranged, the result is,

$$q_h = \frac{2\pi k_h L (T_e - T_i)}{\ln\left(\frac{y_b}{r_e} + \sqrt{(y_b/r_e)^2 - 1}\right)} \quad (12)$$

which defines the steady state heat loss from a buried pipeline as a function of the important parameters.

We would expect that, if the depth of burial was great compared to the pipeline radius, then the log resistance term would look like the well problem solved earlier in these notes, Eq. 2, p. 36. And it does. At this condition y_b/r_e would be considerably greater than 1, so the log term of Eq. 12 would simplify to,

$$\ln(2y_b/r_e) \quad (13)$$

which is identical in form to Eq. 2b of these notes.

It should be of interest to determine when Eq. 12 is necessary and when Eq. 13 is adequate. This can be calculated from the log terms, as shown in the table on the following page.

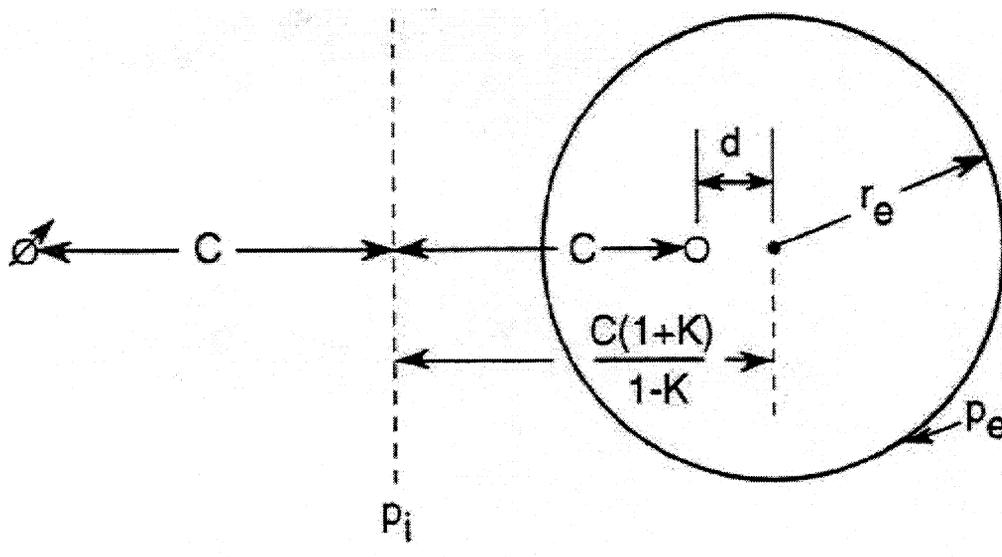
Accuracy of Eq. 13 Compared to Eq. 12

$\frac{y_b}{r_e}$	$\frac{y_b}{r_e} + \sqrt{\left(\frac{y_b}{r_e}\right)^2 - 1}$	$\frac{2y_b}{r_e}$	% Error	$\ln\left[\frac{y_b}{r_e} + \sqrt{\left(\frac{y_b}{r_e}\right)^2 - 1}\right]$	$\ln\left(\frac{2y_b}{r_e}\right)$	% Error
1.5	2.618034	3.0	14.6	0.962424	1.098612	14.2
2.0	3.732051	4.0	7.2	1.316958	1.386294	5.3
2.5	4.791288	5.0	4.4	1.566799	1.609438	2.7
3.0	5.828427	6.0	2.9	1.762747	1.791759	1.6
4.0	7.872983	8.0	1.6	2.063437	2.079442	0.8

Notice that the approximate equation (Eq. 13) is surprisingly accurate even at modest burial depths. This result may not be too surprising when we consider how rapidly the pressures from the point source solution approach those of a finite radius solution as we move away from the wellbore. Of course, these are not quite the same problems, but the distance relations should be expected to behave similarly.

3.3 WELL OFF CENTER IN A CONSTANT PRESSURE CIRCLE

When a constant rate well is being produced from a constant pressure circle, this is the same as a steady state doublet system if we were to look only at the well and one of the constant pressure circles in that system. So a sketch of the system can be shown as follows, with the important variables shown on the sketch.



The careful reader will notice that the geometry of this system is identical to that of the buried pipeline (with x and y interchanged). However, the emphasis is different, for we are interested in the pressure drop from the well to the circle rather than from the circle to the constant, p_i , as we were in the pipeline problem.

To solve this problem, we can write the equation for the pressure drop from the well to the center at p_i , write the equation for the pressure drop from the circle to p_i , and subtract them from each other. The equation for the pressure drop from the well (at C) to p_i is Eq. 2a of these notes, where below I've divided both sides of the equation by two.

$$\frac{2\pi kh(p_{wf} - p_i)}{q\mu} = \ln\left(\frac{r_{wi}}{2C}\right) \quad (2c)$$

The pressure drop from the constant pressure circle to the constant pressure, p_i , is the same in form as Eq. 3, but it must be written in fluid flow terms, and with the signs reversed, since the flow direction is reversed.

$$\frac{2\pi kh(p_e - p_i)}{q\mu} = \frac{1}{2} \ln K = \ln \sqrt{K} \quad (3a)$$

Subtracting Eq. 2c from Eq. 3a, we get,

$$\begin{aligned} \frac{2\pi kh}{q\mu}(p_e - p_{wf}) &= \ln \sqrt{K} + \ln\left(\frac{2C}{r_{wi}}\right) \\ &= \ln\left(\frac{2C\sqrt{K}}{r_{wi}}\right) \end{aligned} \quad (14)$$

Earlier in these notes, we showed that C could be related to r_e and K by Eq. 6.

$$r_e = 2C\sqrt{K}/(1-K) \quad (6)$$

or

$$2C = \frac{r_e(1-K)}{\sqrt{K}} \quad (6a)$$

Thus upon substituting Eq. 6a, the log term in Eq. 14 becomes,

$$\ln\left(\frac{2C\sqrt{K}}{r_{wf}}\right) = -\ln\left[\frac{r_e}{r_{wf}}(1-K)\right] \quad (15a)$$

and K was defined in Eq. 8b in terms of d and r_e . When this definition is substituted into Eq. 15a, it becomes,

$$\ln\left[\frac{r_e}{r_{wf}}(1-K)\right] = \ln\left[\frac{r_e}{r_{wf}}\left(1 - \frac{d^2}{r_e^2}\right)\right] \quad (15b)$$

So the pressure drop equation, from Eq. 14, becomes,

$$q = \frac{2\pi kh}{\mu} \frac{(p_e - p_{wf})}{\ln\left[\frac{r_e}{r_{wf}}\left(1 - \frac{d^2}{r_e^2}\right)\right]} \quad (16)$$

Compare Eq. 16 with Eq. 6.39, p. 299 of Craft and Hawkins (Applied Petroleum Reservoir Engineering). At first glance, they look the same, but they are not. When one looks at their productivity ratio (PR) equation you can see that they treated their equivalent of Eq. 16 as though the term $1 - d^2/r_e^2$ was outside the log rather than inside it. Muskat (Flow of Homogeneous Fluids Through Porous Media," p. 172, Eq. 11) derived the equation and wrote it correctly.

The productivity ratio (PR) as shown by Craft and Hawkins is,

$$PR = \frac{r_e^2}{r_e^2 - d^2} \quad (17)$$

It should be,

$$PR = \frac{\ln(r_e/r_w)}{\ln\left[\frac{r_e}{r_w}\left(1 - \frac{d^2}{r_e^2}\right)\right]} \quad (18)$$

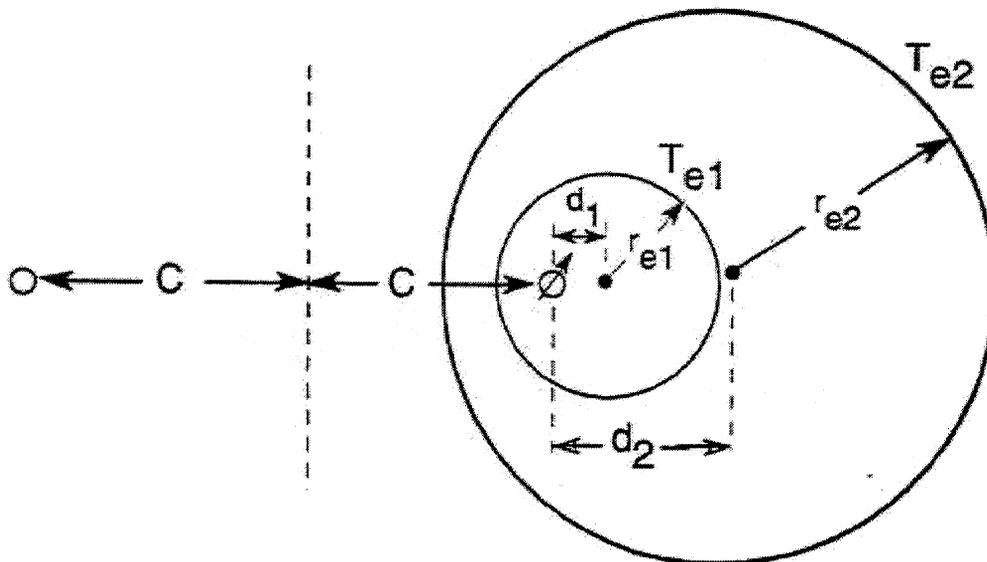
In the sentence below their Eq. 6.39, Craft and Hawkins point out that the effect of having a well off center from the circle is minor. A well 100 ft. off center in a 660 ft. radius circle will have a PR of 1.023 by their calculation using Eq. 17. Actually, that off center effect is even less than they calculate. From Eq. 18, it is clear that the PR depends on the r_e/r_w ratio. The PR will range from 1.005 to 1.0025 as the r_e/r_w ratio ranges from 100 to 10,000.

The off center effect is really negligible for practical purposes. This is both good and bad news. It means that simple equations can be used to calculate production rates even when wells are not centered in the system. That's good news. Unfortunately, it also means that from productivity alone, it is hard to distinguish whether the well is producing from a non-symmetric system. Thus other means are necessary; geologic data, or transient well testing are the most useful.

3.4 HEAT LOSS FROM A STEAM INJECTION WELL

Typically in steam injection wells, the steam is injected down tubing. The annular space between the tubing and casing is often filled with an insulating material to reduce the heat loss rate, and also to protect the casing from the great forces caused by a rise in casing temperature. These forces arise because the hot casing wants to elongate from thermal expansion, and can't be allowed to. There are many documented cases of unprotected thermal wellheads "growing" 5 to 10 feet out of the ground.

Unless special precautions are taken, the tubing will not necessarily hang directly in the center of the casing, thus the heat transfer rate between two off-center cylinders is a problem of practical interest to petroleum engineers. This, too, is a doublet problem as can be seen in the sketch below.



A procedure that can be used to write the equation for the heat flow rate between these two circles is similar to that used before, except now there are two circles at the temperature boundaries of interest. We can write two equations

which include the important parameters of these circles. First, let's write the heat flow equations from an imaginary injection point to the circles. They will be identical in form to Eq. 16 of these notes using heat flow nomenclature and reversing the sign since I've shown an injection well at $x = C$ rather than a producer, as I did in Eq. 16. For Circle 1, the equation is

$$\frac{2\pi k_h L}{q_h} (T_w - T_{e1}) = \ln \left[\frac{r_{e1}}{r_{wi}} \left(1 - \frac{d_1^2}{r_{e1}^2} \right) \right] \quad (19)$$

and for Circle 2, it is

$$\frac{2\pi k_h L}{q_h} (T_w - T_{e2}) = \ln \left[\frac{r_{e2}}{r_{wi}} \left(1 - \frac{d_2^2}{r_{e2}^2} \right) \right] \quad (20)$$

The temperatures of interest are T_{e1} and T_{e2} , which we can solve for by subtracting Eq. 19 from Eq. 20, as follows,

$$\begin{aligned} \frac{2\pi k_h L}{q_h} (T_{e1} - T_{e2}) &= \ln \frac{[r_{e2}(1 - d_2^2/r_{e2}^2)]}{[r_{e1}(1 - d_1^2/r_{e1}^2)]} \\ &= \ln \left[\frac{r_{e1}(r_{e2}^2 - d_2^2)}{r_{e2}(r_{e1}^2 - d_1^2)} \right] \end{aligned} \quad (21)$$

Equation 21 gives temperature drop as a function of the important parameters. But it contains two variables, d_1 and d_2 , and it is only one equation. Another equation is needed. The obvious one to consider is the equation for the temperature drop from the circles to the center of the system, which will be the same form as Eq. 3 of these notes. When Eq. 3 is written in heat transfer terms for the two circles of this system, it becomes

$$\frac{2\pi k_h L}{q_h} (T_{e1} - T_i) = \ln \sqrt{1/K_1} = \ln \left(\frac{r_{e1}}{d_1} \right) \quad (22)$$

and

$$\frac{2\pi k_h L}{q_h} (T_{e2} - T_i) = \ln \sqrt{1/K_2} = \ln \left(\frac{r_{e2}}{d_2} \right) \quad (23)$$

Note that, in Eqs. 22 and 23, I have invoked the fact that $\sqrt{K} = d/r_e$, from Eq. 8 of these notes.

Now we can solve for the temperature drop between the circles by subtracting Eq. 23 from Eq. 22, as seen below,

$$\frac{2\pi k_h L}{qh}(T_{e1} - T_{e2}) = \ln\left(\frac{r_{e1} d_2}{r_{e2} d_1}\right) \quad (24)$$

Note that the left hand sides of Eqs. 21 and 24 are identical. Thus the log terms must also be identical. This leads to,

$$\ln\left[\frac{r_{e1}}{r_{e2}}\left(\frac{r_{e2}^2 - d_2^2}{r_{e1}^2 - d_1^2}\right)\right] = \ln\left(\frac{r_{e1} d_2}{r_{e2} d_1}\right) \quad (25a)$$

or

$$\frac{r_{e2}^2 - d_2^2}{r_{e1}^2 - d_1^2} = \frac{d_2}{d_1} \quad (25b)$$

Thus we have an equation that directly relates d_1 to d_2 . A further relation comes from defining the distance off center for the tubing inside the casing, which we would expect to be an important parameter in the heat transfer process. Calling the distance r_{oc} , from the sketch of the system it is obvious that,

$$d_2 = r_{oc} + d_1 \quad (26)$$

This definition can be substituted into Eq. 25b resulting in an equation for d_1 as a function of the important parameters, r_{e2} , r_{e1} and r_{oc} . After considerable algebra, the equation becomes,

$$d_1^2 - \frac{(r_{e2}^2 - r_{e1}^2 - r_{oc}^2)}{2r_{oc}}d_1 + r_{e1}^2 = 0 \quad (27)$$

This is a quadratic equation in d_1 . For the parameters of interest to us, the negative square root term is the one to use.

So the solution is,

$$d_1 = \frac{(r_{e2}^2 - r_{e1}^2 - r_{oc}^2) - \sqrt{(r_{e2}^2 - r_{e1}^2 - r_{oc}^2)^2 - 4r_{e1}^2 r_{oc}^2}}{2r_{oc}} \quad (28a)$$

$$= \frac{r_{e2}^2 - r_{e1}^2 - r_{oc}^2 - \sqrt{r_{e2}^4 + r_{e1}^4 + r_{oc}^4 - 2r_{e2}^2 r_{e1}^2 - 2r_{oc}^2 - 2r_{oc}^2 (r_{e1}^2 + r_{e2}^2)}}{2r_{oc}} \quad (28b)$$

From Eq. 26 we get,

$$d_2 = r_{oc} + d_1 = \frac{2r_{oc}^2}{2r_{oc}} + d_1 \quad (29a)$$

and when substituting Eq. 28b into Eq. 29a we get,

$$d_2 = \frac{\left(r_{e2}^2 - r_{e1}^2 + r_{oc}^2\right) - \sqrt{r_{e2}^4 + r_{e1}^4 + r_{oc}^4 - 2r_{e2}^2 r_{e1}^2 - 2r_{oc}^2 \left(r_{e1}^2 + r_{e2}^2\right)}}{2r_{oc}} \quad (29b)$$

Notice that the only difference between Eq. 29b and 28b is the sign on the r_{oc}^2 term outside the square root.

Equations 28b and 29b can be substituted into the argument of the log term in Eq. 24 to get,

$$\frac{r_{e1}d_2}{r_{e2}d_1} = \left(\frac{r_{e1}}{r_{e2}}\right) \left[\frac{r_{e2}^2 - r_{e1}^2 + r_{oc}^2 - \sqrt{r_{e2}^4 + r_{e1}^4 + r_{oc}^4 - 2r_{e2}^2 r_{e1}^2 - 2r_{oc}^2 \left(r_{e2}^2 + r_{e1}^2\right)}}{r_{e2}^2 - r_{e1}^2 - r_{oc}^2 - \sqrt{r_{e2}^4 + r_{e1}^4 + r_{oc}^4 - 2r_{e2}^2 r_{e1}^2 - 2r_{oc}^2 \left(r_{e2}^2 + r_{e1}^2\right)}} \right] \quad (30)$$

For calculational convenience, Eq. 30 can be divided top and bottom by r_{e2}^2 , as follows,

$$\frac{r_{e1}d_2}{r_{e2}d_1} = \left(\frac{r_{e1}}{r_{e2}}\right) \left\{ \frac{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 + \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \sqrt{1 + \left(\frac{r_{e1}}{r_{e2}}\right)^4 + \left(\frac{r_{oc}}{r_{e2}}\right)^4 - 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - 2\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[1 + \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]}}{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \sqrt{1 + \left(\frac{r_{e1}}{r_{e2}}\right)^4 + \left(\frac{r_{oc}}{r_{e2}}\right)^4 - 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - 2\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[1 + \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]}} \right\} \quad (31a)$$

or, after some further simple algebra,

$$\frac{r_{e1}d_2}{r_{e2}d_1} = \left(\frac{r_{e1}}{r_{e2}}\right) \left\{ \frac{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 + \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \sqrt{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}}{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \sqrt{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}} \right\} \quad (31b)$$

Equation 31b is the log term for the heat flow equation, Eq. 21 or 24. So the heat flow rate is,

$$q_h = \frac{2\pi k_h L (T_{e1} - T_{e2})}{\ln\left(\frac{r_{e1}}{r_{e2}}\right) \left\{ \frac{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 + \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \sqrt{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}}{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \sqrt{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}} \right\}} \quad (32)$$

3.4.1 Limits at Boundaries

It is important to find out whether Eqs. 31b and 32 properly degenerate to the correct equation at the limits as r_{oc} ranges from zero to $r_{e2} - r_{e1}$. When r_{oc} is zero, the log term (Eq. 31b) should degenerate to r_{e2}/r_{e1} , the resistance term when the tubing is in the center of the casing. At the other limit, when $r_{oc} = r_{e2} - r_{e1}$, the tubing is touching the casing, and there is no resistance to heat flow, so the log term should be exactly equal to 1. 0.

Let's first look at the log term when $r_{oc} = 0$. Equation 31b then simplifies to,

$$\begin{aligned} \frac{r_{e1}}{r_{e2}} \frac{d_2}{d_1} &= \left(\frac{r_{e1}}{r_{e2}}\right) \left\{ \frac{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \sqrt{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2 - 0}}{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \sqrt{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2 - 0}} \right\} \\ &= \left(\frac{r_{e1}}{r_{e2}}\right) \left(\frac{0}{0}\right) \end{aligned} \quad (33)$$

Thus Eq. 31b becomes indeterminate. But it can be solved by looking at the terms in Eq. 31b as r_{oc}/r_{e2} approaches zero. First we will look at the square root term in both the numerator and denominator. At the limit, the ratio r_{oc}/r_{e2}

is small compared to $1 - (r_{e1}/r_{e2})^2$. Thus we can rearrange the square root term as follows,

$$\sqrt{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]} = \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] \sqrt{1 - \frac{\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2}} \quad (34)$$

Since under this condition $(r_{oc}/r_{e2})^2$ is very small compared to 1.0, this square root term can be treated as follows,

$$\sqrt{1-a} \approx 1 - a/2 \quad (35)$$

to become,

$$\sqrt{1 - \frac{\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2}} \approx 1 - \frac{\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}{2 \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2} \quad (36a)$$

Since $(r_{oc}/r_{e2})^2$ is very small compared to the other two terms, in the numerator bracket on the right-hand side, Eq.

36a simplifies to,

$$\sqrt{1 - \frac{\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[2 + 2\left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2}} \approx 1 - \frac{\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[1 + \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]}{\left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]^2} \quad (36b)$$

Equation 36b can be substituted into Eq. 34 and that, in turn, can be substituted into Eq. 31b to get,

$$\frac{r_{e1}d_2}{r_{e2}d_2} = \left(\frac{r_{e1}}{r_{e2}}\right) \left\{ \frac{\left\{ 1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 + \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] \left[1 - \frac{(r_{oc}/r_{e2})^2 [1 + (r_{e1}/r_{e2})^2]}{[1 - (r_{e1}/r_{e2})^2]^2}\right]}{\left\{ 1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] \left[1 - \frac{(r_{oc}/r_{e2})^2 [1 + (r_{e1}r_{e2})^2]}{[1 - (r_{e1}r_{e2})^2]^2}\right]} \right\}}{\left\{ 1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] \left[1 - \frac{(r_{oc}/r_{e2})^2 [1 + (r_{e1}r_{e2})^2]}{[1 - (r_{e1}r_{e2})^2]^2}\right]} \right\}} \quad (37a)$$

which simplifies to,

$$\frac{r_{e1}d_2}{r_{e2}d_2} = \left(\frac{r_{e1}}{r_{e2}}\right) \frac{\left\{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 + \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] + \left(\frac{r_{oc}}{r_{e2}}\right)^2 \frac{[1 + (r_{e1}/r_{e2})^2]}{[1 - (r_{e1}/r_{e2})^2]}\right\}}{\left\{1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2 - \left(\frac{r_{oc}}{r_{e2}}\right)^2 - \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] + \left(\frac{r_{oc}}{r_{e2}}\right)^2 \frac{[1 + (r_{e1}/r_{e2})^2]}{[1 - (r_{e1}/r_{e2})^2]}\right\}} \quad (37b)$$

or,

$$\frac{r_{e1}d_2}{r_{e2}d_1} = \left(\frac{r_{e1}}{r_{e2}}\right) \frac{\left\{\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] + \left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[1 + \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]\right\}}{\left\{-\left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[1 - \left(\frac{r_{e1}}{r_{e2}}\right)^2\right] + \left(\frac{r_{oc}}{r_{e2}}\right)^2 \left[1 + \left(\frac{r_{e1}}{r_{e2}}\right)^2\right]\right\}} \quad (37c)$$

or,

$$\frac{r_{e1}d_2}{r_{e2}d_1} = \left(\frac{r_{e1}}{r_{e2}}\right) \frac{\left[2\left(\frac{r_{oc}}{r_{e2}}\right)^2\right]}{\left[2\left(\frac{r_{oc}}{r_{e2}}\right)^2\right] \left[\left(\frac{r_{e1}}{r_{e2}}\right)^2\right]} = \left(\frac{r_{e1}}{r_{e2}}\right) \left(\frac{r_{e2}}{r_{e1}}\right)^2 = \frac{r_{e2}}{r_{e1}} \quad (37d)$$

Thus Eq. 31b properly degenerates to the ratio, r_{e2}/r_{e1} , as it should when $r_{oc} = 0$, that is, when the tubing is in the center of the casing.

Now let's consider Eq. 31b when the tubing lies against the casing. Under that circumstance, the following equation holds,

$$r_{oc} = r_2 - r_1 \quad (38a)$$

$$\text{or,} \quad \frac{r_{oc}}{r_{e2}} = 1 - \frac{r_{e1}}{r_{e2}} \quad (38b)$$

$$\text{or, } \left(\frac{r_{oc}}{r_{e2}} \right)^2 = 1 - \frac{2r_{e1}}{r_{e2}} + \left(\frac{r_{e1}}{r_{e2}} \right)^2 \quad (38c)$$

We can substitute Eq. 38c into the square root terms of Eq. 31b, as follows,

$$\left[1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 \right]^2 - \left(\frac{r_{oc}}{r_{e2}} \right)^2 \left[2 + 2 \left(\frac{r_{e1}}{r_{e2}} \right)^2 - \left(\frac{r_{oc}}{r_{e2}} \right)^2 \right] = \left[1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 \right]^2 - \left[1 - \frac{r_{e1}}{r_{e2}} \right]^2 \left[2 + 2 \left(\frac{r_{e1}}{r_{e2}} \right)^2 - 1 + \frac{2r_{e1}}{r_{e2}} - \left(\frac{r_{e1}}{r_{e2}} \right)^2 \right] \quad (39a)$$

$$= \left[1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 \right]^2 - \left[1 - \frac{r_{e1}}{r_{e2}} \right]^2 \left[1 + \frac{2r_{e1}}{r_{e2}} + \left(\frac{r_{e1}}{r_{e2}} \right)^2 \right] \quad (39b)$$

$$= \left[1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 \right]^2 - \left[1 - \frac{r_{e1}}{r_{e2}} \right]^2 \left[1 + \frac{r_{e1}}{r_{e2}} \right]^2 = 0 \quad (39c)$$

Equation 39c shows that the square root terms in Eq. 31b are zero when the tubing lays along the casing. Thus Eq. 31b simplifies to,

$$\frac{r_{e1}d_2}{r_{e2}d_1} = \left(\frac{r_{e1}}{r_{e2}} \right) \left[\frac{1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 + \left(\frac{r_{oc}}{r_{e2}} \right)^2}{1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 - \left(\frac{r_{oc}}{r_{e2}} \right)^2} \right] \quad (40a)$$

and when Eq. 38c is substituted into Eq. 40a, it becomes,

$$\frac{r_{e1}d_2}{r_{e2}d_1} = \left(\frac{r_{e1}}{r_{e2}} \right) \left[\frac{1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 + 1 - 2 \left(\frac{r_{e1}}{r_{e2}} \right) + \left(\frac{r_{e1}}{r_{e2}} \right)^2}{1 - \left(\frac{r_{e1}}{r_{e2}} \right)^2 - 1 + 2 \left(\frac{r_{e1}}{r_{e2}} \right) - \left(\frac{r_{e1}}{r_{e2}} \right)^2} \right] \quad (40b)$$

$$= \left(\frac{r_{e1}}{r_{e2}} \right) \frac{\left[2 - 2 \left(\frac{r_{e1}}{r_{e2}} \right) \right]}{\left[2 \left(\frac{r_{e1}}{r_{e2}} \right) - 2 \left(\frac{r_{e1}}{r_{e2}} \right)^2 \right]} = \left(\frac{r_{e1}}{r_{e2}} \right) \frac{1}{\left(\frac{r_{e1}}{r_{e2}} \right)} = 1 \quad (40c)$$

Thus the equation properly approaches the limit of 1.0 as the tubing approaches the casing wall.

3.4.2 Final Heat Transfer Results

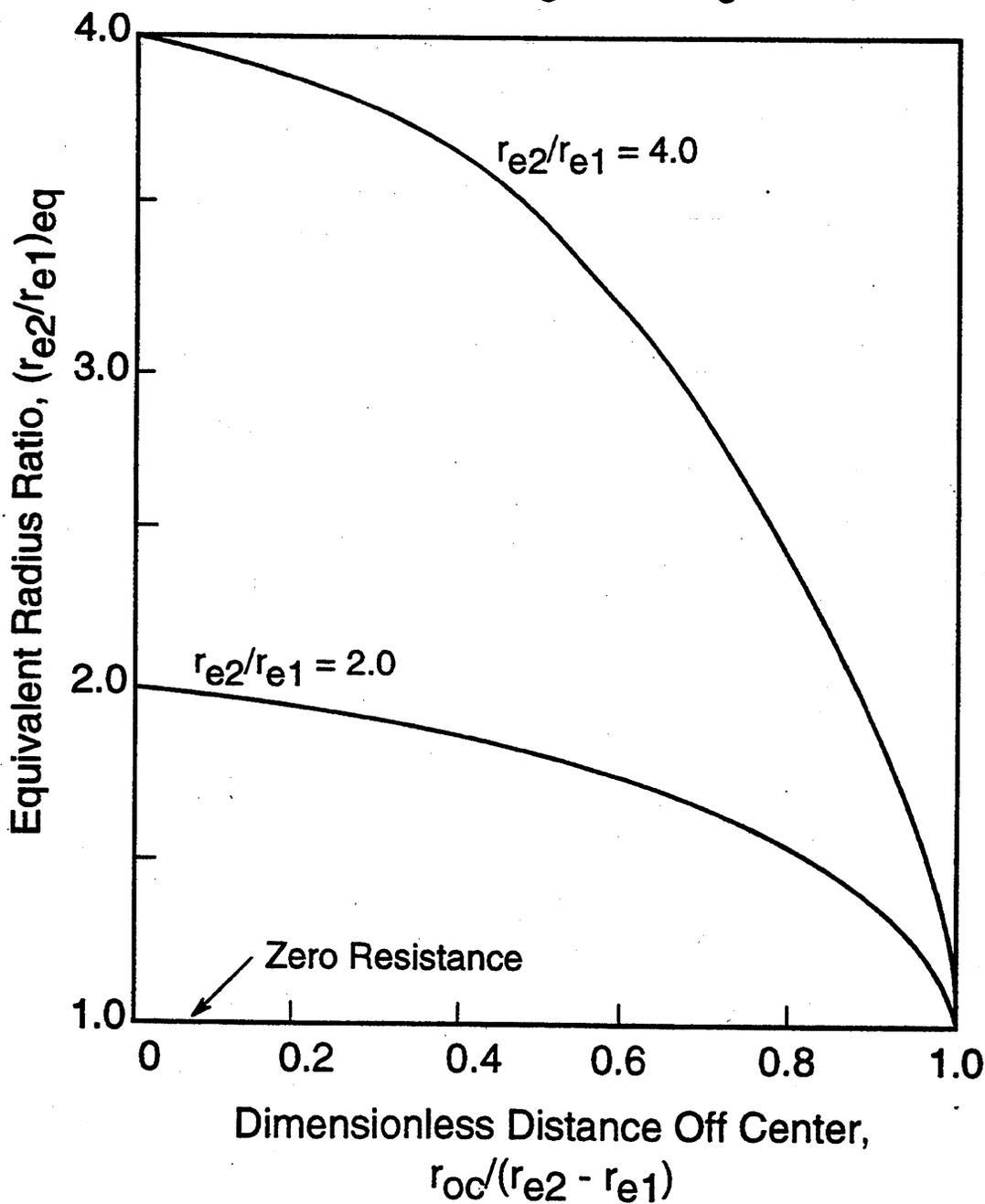
Clearly, Eq. 32 gives the correct expression for the heat transfer rate between the two off center cylinders. It would be useful to find out how the log resistance term varies as the tubing moves off center in the casing. To look at this in detail, I will consider two cases: one will be for $r_{e2} / r_{e1} = 4.0$ and the other for $r_{e2} / r_{e1} = 2.0$. This is about as broad a range as we would normally see in steam injection wells. I'll not bother to detail the calculations involved. They are straightforward, though a bit tedious. The results are shown in the table below and the graph on the next page. Additional approximate results, which will be discussed later, are also included in this table.

Equivalent Radius Ratios Exact Values and Approximate Values from Eq. 41

$r_{e2} / r_{e1} = 2.0$			$r_{e2} / r_{e1} = 4.$		
r_{oc} / r_{e2}	Equiv. r ratios $(r_{e2} / r_{e1})_{eq.}$	% Error Eq. 41	r_{oc} / r_{e2}	Equiv. r ratios $(r_{e2} / r_{e1})_{eq.}$	% Error Eq. 41
0	2.0000	0.00	0.00	4.0000	0.00
0.0625	1.9896	-0.11	0.10	3.9573	-0.24
0.1250	1.9580	-0.16	0.20	3.8288	-0.38
0.1875	1.9047	-0.09	0.30	3.6132	-0.30
0.2500	1.8279	0.08	0.40	3.3077	-0.02
0.3125	1.7250	0.30	0.50	2.9059	0.41
0.3750	1.5897	0.48	0.60	2.3919	0.81
0.4375	1.4059	0.40	0.70	1.7027	0.45
0.46875	1.2782	0.06	0.725	1.4860	-0.14
0.484375	1.1914	-0.28	0.7375	1.3124	-0.65
0.4921875	1.1325	-0.52	0.74375	1.2128	-0.97
0.5000	1.0000	0.00	0.75	1.0000	0.00

Notice the ends of the columns in this table. There are more values listed as the tubing approaches the casing wall. These are there to better define the drop in resistance under this circumstance. It is clear from the results in this table, and the graph below, that the off-center effect is quite modest until the tubing has moved approximately halfway toward the wall. Thereafter the resistance drops rapidly, particularly during the last fifteen percent, or so, of its possible movement.

Equivalent Radius Ratio for Off Center Tubing in Casing



The message these results give us is that the off-center tubing effect can be serious if the tubing is lying on or very near the casing, but it is not necessary that it be exactly centered in the system. Clearly, however, some means should be used to keep the tubing away from the casing inner surface.

3.4.3 Approximation to Equations 31 and 32

Equations 31 and 32 give us the means to calculate heat transfer between tubing and casing when the tubing is not centered. It is clear, from the table on page 52, that the equivalent radius ratio ranges from the actual value of r_{e2}/r_{e1} to 1.00 as the tubing moves off center. Equation 31, which is the argument of the log term in Eq. 32, is quite complex, however; it is a pain to solve. Realizing this, I have developed an empirical equation to approximate Eq. 31, the argument of the log term, denoted $(r_{e2}/r_{e1})_{eq}$ in the equation. It too, is a bit complex, but is considerably simpler than Eqs. 31 and 32. It is,

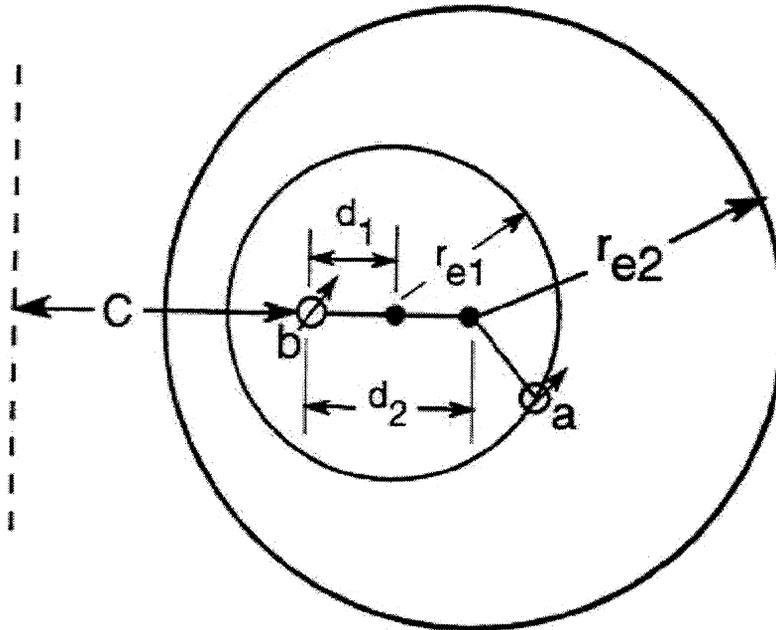
$$\left[\frac{\left(\frac{r_{e2}}{r_{e1}}\right)^{1/4} - \left(\frac{r_{e2}}{r_{e1}}\right)^{1/4}}{1 - \left(\frac{r_{e2}}{r_{e1}}\right)^{1/4}} \right]^{0.568} = 1 - \left[\frac{\frac{r_{e2}}{r_{e1}} - 1 - \frac{r_{oc}}{r_{e1}}}{\left(\frac{r_{e2}}{r_{e1}}\right) - 1} \right]^{0.568} \quad (41)$$

I have tested this equation for r_{e2}/r_{e1} ranging from 1.5 to 5.0, and for r_{oc} ranging from 0 to $r_{e2} - r_{e1}$. This is a broader range than we would generally expect to see in wells, but the equation seems to hold well over this entire range. The maximum error calculated in $(r_{e2}/r_{e1})_{eq}$ using this equation was less than 1%. In the table on p. 52, I've shown a listing of these errors for r_{e2}/r_{e1} equal to 2.0 and 4.0. To use this equation (Eq. 41), the calculated value of $(r_{e2}/r_{e1})_{eq}$ replaces Eq. 31 and is substituted into the log term of Eq. 32.

3.5 ADDITIONAL WELLS IN A CONSTANT PRESSURE CIRCLE

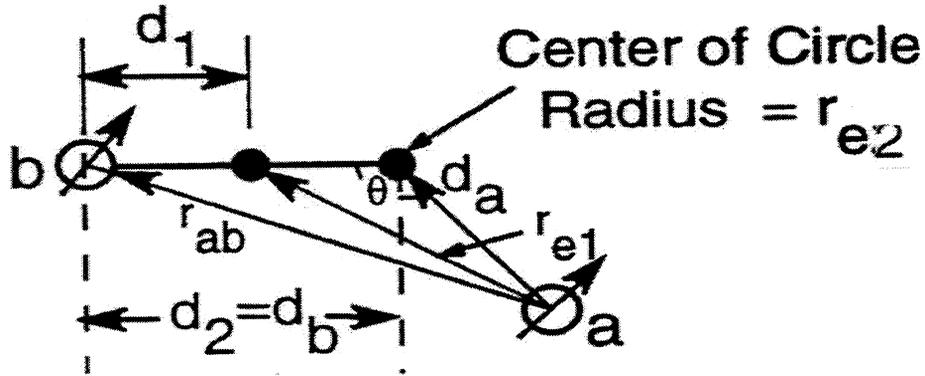
If there are two or more wells injecting into, or producing from, a constant pressure circle, the pressure at each well is a simple superposition of the effect of that well plus the interference effect of the offset wells. Thus it is important to determine that interference effect. It turns out that the geometry and mathematics of this interference

effect is similar to the off-center radial heat flow problem we just solved. Let me show this idea in the following diagrams.



In the above sketch, I've shown Well a off center in the large circle whose radius is r_{e2} . Well b is another well in that circle which is also off center. We are interested in the pressure effect of Well b on Well a. As far as Well b is concerned, Well a could be anywhere in the circle defined by d_1 and r_{e1} . The interference effect will be the same anywhere on that circle, so we need to define the pressure there. For our equivalent heat transfer problem we had defined the temperature drops in terms of the two radii and the distance off center. For this problem, we need to define the pressure drops in terms of the distance between the wells and their distances from the center of Circle 2. Thus, although the problem is the same, the terms that we need to solve for have changed. An enlarged sketch of this idea is shown on the following page, with the important variables labeled.

Notice that I've shown Well a at any general angle, θ , on the circle whose radius is r_{e1} . The important distance variables between the wells are d_a , d_b and r_{ab} , so our pressure drop should be stated in terms of these variables plus the overall radius of our constant pressure system, r_{e2} .



The distance relations, however, contain the general angle, θ . This term can be eliminated as follows. First we write the general equation that relates r_{ab} , d_a , d_b , and angle θ on the larger triangle above.

$$r_{ab}^2 = d_a^2 + d_b^2 - 2d_a d_b \cos \theta \quad (42a)$$

or,

$$\cos \theta = \frac{d_a^2 + d_b^2 - r_{ab}^2}{2d_a d_b} \quad (42b)$$

We can also write the equation for the inner triangle in the sketch above,

$$r_{e1}^2 = (d_b - d_1)^2 + d_a^2 - 2(d_b - d_1)(d_a) \cos \theta \quad (43a)$$

and substitute Eq. 42b,

$$\begin{aligned} r_{e1}^2 &= (d_b - d_1)^2 + d_a^2 - \frac{(d_b - d_1)(d_a^2 + d_b^2 - r_{ab}^2)}{d_b} \\ &= \frac{(d_b - d_1)^2 d_b + d_a^2 d_b - (d_b - d_1)(d_a^2 + d_b^2 - r_{ab}^2)}{d_b} \end{aligned} \quad (43b)$$

In my earlier notes on the off-center radial heat transfer problem in Eq. 25, p. 46, I calculated the radial resistance factors two different ways so that r_{e1} , r_{e2} , d_1 and d_2 could be related to each other. The result was,

$$\frac{r_{e2}^2 - d_2^2}{r_{e1}^2 - d_1^2} = \frac{d_2}{d_1} \quad (25b)$$

I will now rearrange this equation in a different way to solve for $(r_{e1} / d_1)^2$,

$$\left(\frac{r_{e1}}{d_1}\right)^2 - 1 = \frac{r_{e2}^2 - d_2^2}{d_1 d_2} \quad (44a)$$

or,

$$\left(\frac{r_{e1}}{d_1}\right)^2 = \frac{r_{e2}^2 - d_2^2 + d_1 d_2}{d_1 d_2} \quad (44b)$$

and since $d_2 = d_b$,

$$\left(\frac{r_{e1}}{d_1}\right)^2 = \frac{r_{e2}^2 - d_b^2 + d_1 d_b}{d_1 d_b} \quad (44c)$$

Equation 43b can also be put to terms of $(r_{e1} / d_1)^2$,

$$\begin{aligned} \left(\frac{r_{e1}}{d_1}\right)^2 &= \frac{(d_b - d_1)^2 d_b + d_a^2 d_b - (d_b - d_1)(d_a^2 + d_b^2 - r_{ab}^2)}{d_b d_1^2} \\ &= \frac{d_a^2 d_1 - d_b^2 d_1 + d_b d_1^2 + d_b d_1^2 + d_b r_{ab}^2 - d_1 r_{ab}^2}{d_b d_1^2} \end{aligned} \quad (43c, d)$$

Now Eqs. 44c and 43c can be set equal to each other, with the following result,

$$r_{e2}^2 - d_b^2 + d_b d_1 = \frac{d_a^2 d_1 - d_b^2 d_1 + d_b d_1^2 + d_b r_{ab}^2 - d_1 r_{ab}^2}{d_1} \quad (45a)$$

or, upon rearranging,

$$r_{e2}^2 d_1 - d_b^2 d_1 + d_b d_1^2 = d_a^2 d_1 - d_b^2 d_1 + d_b d_1^2 + d_b r_{ab}^2 - d_1 r_{ab}^2 \quad (45b)$$

or,

$$d_1(r_{e2}^2 - d_a^2 + r_{ab}^2) = d_b r_{ab}^2 \quad (45c)$$

or,

$$d_1 d_b = \frac{d_b^2 r_{ab}^2}{r_{e2}^2 - d_a^2 + r_{ab}^2} \quad (45d)$$

Thus we have written an equation which relates d_1 , the unwanted term, to variables we do want in our final equation.

Now Eq. 45d can be substituted into Eq. 44c with the following result,

$$\left(\frac{r_{e1}}{d_1}\right)^2 = \frac{(r_{e2}^2 - d_b^2)(r_{e2}^2 - d_a^2 + r_{ab}^2) + d_b^2 r_{ab}^2}{d_b^2 r_{ab}^2} \quad (46a)$$

or,

$$\left(\frac{r_{e1}}{d_1}\right)^2 = \frac{r_{e2}^4 - d_a^2 r_{e2}^2 + r_{ab}^2 r_{e2}^2 - d_b^2 r_{e2}^2 + d_a^2 d_b^2}{d_b^2 r_{ab}^2} \quad (46b)$$

The argument of the log resistance term shown in Eq. 24, p. 46, is $r_{e1} d_2 / r_{e2} d_1$, but it would be more convenient to square the term inside the log, and substitute Eq. 46b directly.

$$\left(\frac{r_{e1}}{d_1}\right)^2 \left(\frac{d_2}{r_{e2}}\right)^2 = \left(\frac{d_b}{r_{e2}}\right)^2 \left[\frac{r_{e2}^4 - d_a^2 r_{e2}^2 + r_{ab}^2 r_{e2}^2 - d_b^2 r_{e2}^2 + d_a^2 d_b^2}{d_b^2 r_{ab}^2} \right] \quad (47a)$$

$$= \frac{d_a^2 d_b^2 + r_{e2}^2 (r_{e2}^2 - d_a^2 - d_b^2 + r_{ab}^2)}{r_{e2}^2 r_{ab}^2} \quad (47b)$$

Equation 47b is the argument of the log term defining the interference effect of Well b on Well a.

Thus the total pressure increase at Well a due to injection into both Well a and Well b becomes the sum of the effects from both wells, as follows,

$$\frac{2\pi kh}{\mu}(p_{wa} - p_e) = q_a \ln \left[\frac{r_{e2}}{r_{wa}} \left(1 - \frac{d_a^2}{r_{e2}^2} \right) \right] + \frac{q_b}{2} \ln \left[\frac{d_a^2 d_b^2 + r_{e2}^2 (r_{e2}^2 - d_a^2 - d_b^2 + r_{ab}^2)}{r_{e2}^2 r_{ab}^2} \right] \quad (48)$$

The first term on the right of Eq. 48 is Eq. 16, p. 43; the effect of Well a alone. The second term is the effect of Well b on Well a from Eqs. 24 and 47b. Note that the multiplier on the second log is $\frac{1}{2}$. This is because I squared the distance terms in Eqs. 46 and 47.

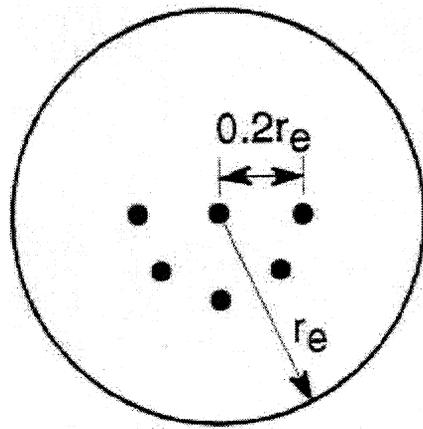
It is interesting that Eq. 48 is not the same as stated by Muskat (Flow of Homogeneous Fluids Through Porous Media, p. 513, Eq. 3). Muskat's equation, slightly changed to fit the nomenclature used here and including differing rates in the wells, is,

$$\frac{2\pi kh}{\mu}(p_{wa} - p_e) = q_a \ln \left(\frac{r_e}{r_{wa}} \right) + q_b \ln \left(\frac{r_e}{r_{ab}} \right) \quad (49)$$

What Muskat assumed to derive this equation was that the outer boundary pressure at r_{e2} , did not vary much over its perimeter, so an average pressure calculated from Eq. 49 could be used. This assumption becomes more valid as the outer boundary becomes further removed from the wells. Thus when the distances d_2 , d_b , and r_{ab} are small compared to r_{e2} , his assumption becomes more correct. It is also clear that Eq. 48 degenerates to Eq. 49 under these conditions.

It should be of interest to see what effect Muskat's assumption has on the resulting pressure drop calculation using his Eq. 49 compared to the much more complex log terms of Eq. 48. The effect of errors in the first log term has already been discussed previously, immediately following Eq. 18, p. 43 of these notes. The effect was found to be modest.

We need also to look at the effect of the second complex log term compared to Muskat's simple expression. To test this, I have looked at two wells that are a distance $0.2 r_e$ from the center of the constant pressure circle. They will be at varying angles from each other as shown in the following sketch, at angles of 45° , 90° , 135° and 180° .



The resulting fixed geometric relations are as follows,

$$d_a = d_b = 0.2r_e \quad (50)$$

$$d_a^2 d_b^2 = 0.0016r_e^4 \quad (51)$$

and the following table shows the magnitude of the errors.

Exact Equation Compared to muscat's Values

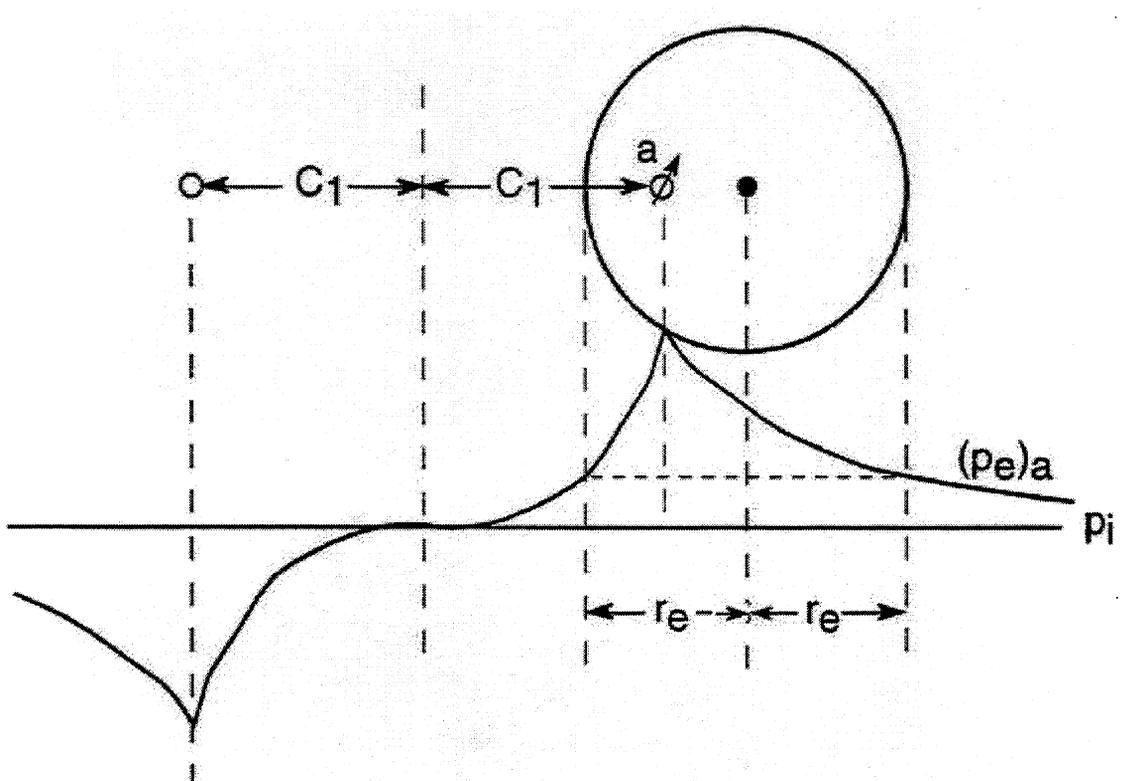
Angle	$1 - \frac{d_a^2 + d_b^2 - r_{ab}^2}{r_e^2}$	Eq. 47b (Multiplier on r_e^2 / r_{ab}^2)	Equivalent Radius Ratio $(r_e / r_{ab})_{eq.}$	Error in Muskat Eq. %
45°	0.943431	0.945031	0.972127	+2.87%
90°	1.000000	1.001600	1.000800	-0.08%
135°	1.056569	1.058169	1.028673	-2.79%
180°	1.080000	1.081600	1.0400000	-3.85%

This table shows that the wells can be rather widely spaced away from the center of the circle, and Muskat's formula is reasonably accurate. Remember also that the actual error in pressure depends on the log of the terms - not the terms themselves; so the effect of the slight errors is even smaller than this table indicates. Further, it should be remembered that the interference effect is always modest compared to the effect of the well itself. Thus the first term on the right hand side of Eqs. 48 and 49 is always considerably greater than the second term.

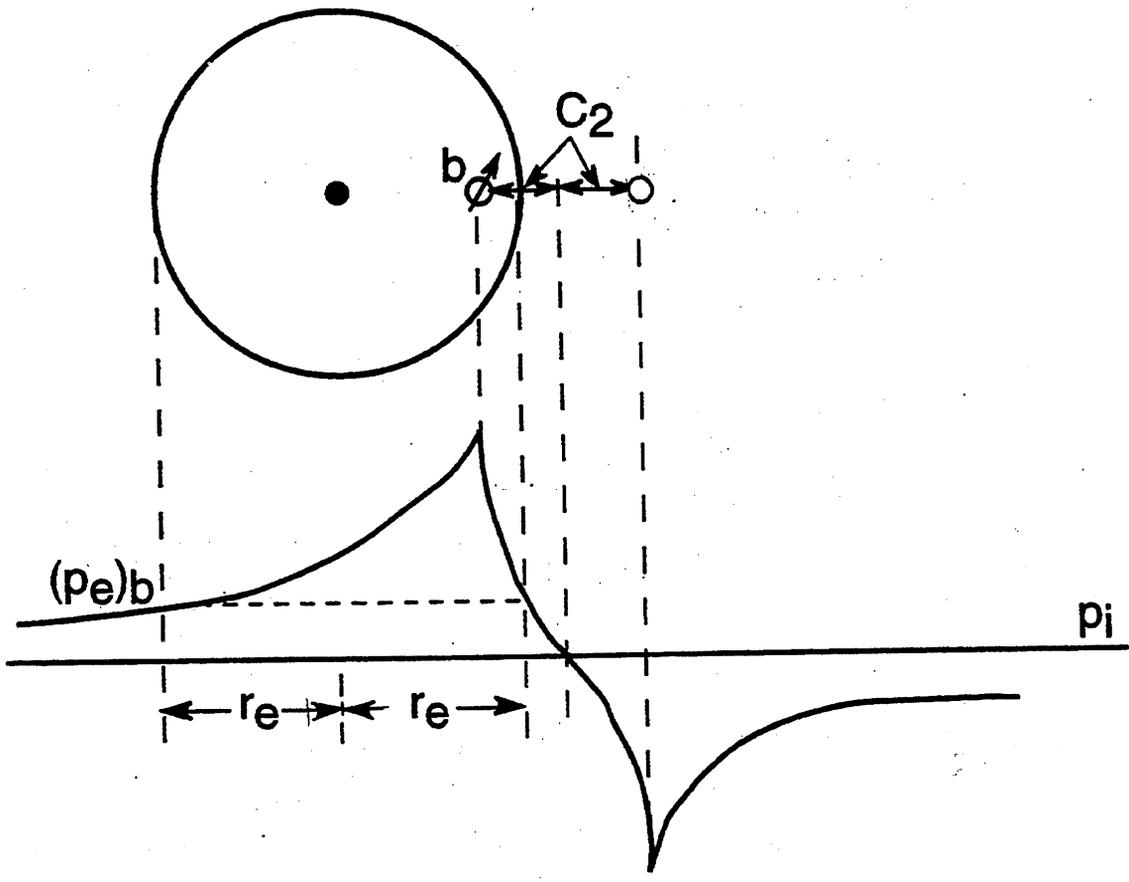
3.5.1 Pressure Fields

It might be of interest to see how the superposed pressure fields look on a graph. Such a graph can only be conveniently drawn if the wells lie along the common diameter at zero y and the resulting pressures are graphed along that same diameter. Such a series of sketches is shown below.

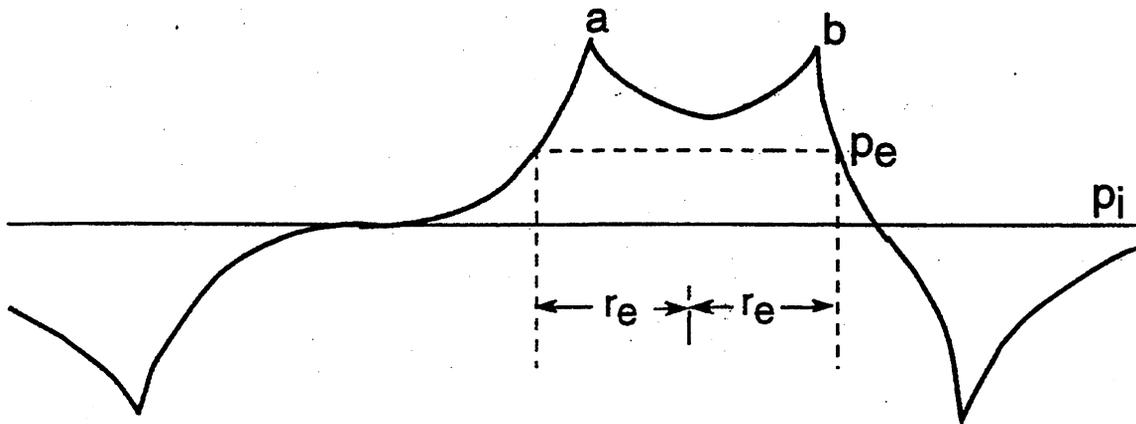
First, let us draw the pressure field for one well and its image as we scan in x along the y axis. Note the horizontal dashed line in this sketch which indicates the pressure on the circumference of the circle. It is labeled, $(p_e)_a$,



Next we'll draw the second well (Well b) and its image. I've made this well nearer to the right edge of the circle, on the other side of it. Again, the constant pressure circle is shown as a horizontal dashed line labeled $(p_e)_b$.



When the effects of the two wells are added together (superposition), the result is as shown on the sketch on the following page.



Notice that the resulting pressure field calculation is very complex indeed; but that doesn't matter. The important point is that the circle of interest labeled, p_e , retains constant pressure at its circumference. Thus the pressure field inside the circle has its correct shape.

Clearly Eq. 48, or alternatively Eq. 49, can be easily expanded to handle superposition whenever there are more than two wells in a circle. It is important to realize that the equations also will handle any combination of injectors and producers, at any rates and at any locations.

4. ISOLATED PATTERNS

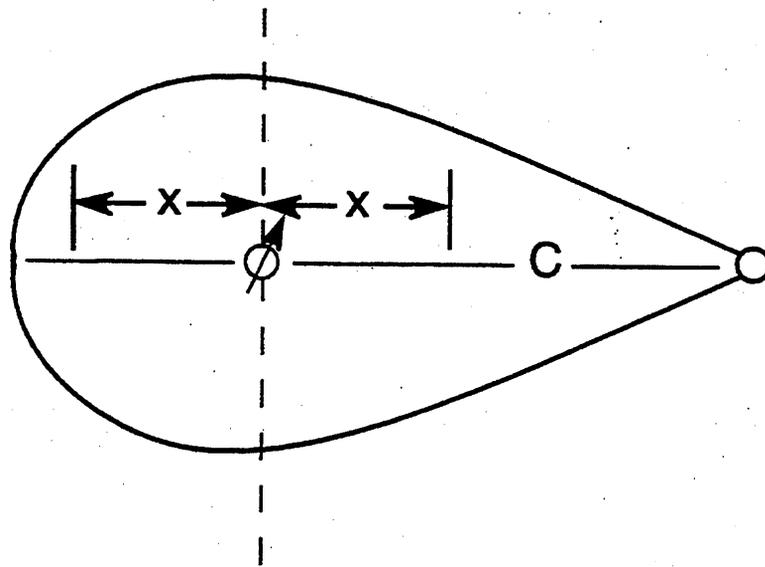
We have been discussing the doublet system, at a mobility ratio of unity, in some detail: the shapes of the constant pressure and flow lines, the areal sweep and breakthrough behavior, the locations of the flood fronts as the displacing fluid moves in the reservoir, and various practical heat and fluid flow problems that can be solved using doublet geometry. The analytic solutions we have been using are not limited to a two-well system. They can be invoked any time the total injection and production rates are equal--as will be seen in the material presented herein. A number of interesting multiple well patterns will be addressed.

4.1 PILOT FLOODING PATTERNS

Here, I'm going to look at several isolated patterns whose total injection and production are equal. We will find that there is a sequence of results from those patterns that makes it obvious what can happen when the number of wells is increased. First we will look again at the two-spot (the doublet) in a slightly different way than before.

4.1.1. The Isolated Two-Spot (Doublet)

I've already shown the behavior of the two-spot in earlier notes (Sections 1, 2, and 3). Here I'll change the distance definitions slightly for reasons that will become clear when the later, more complex, patterns are addressed. A sketch of the system is shown below.



Notice, in this sketch, that the injector is shown at the origin. The distance between the injector and producer is labeled C rather than $2C$ as it was in the earlier notes.

From the sketch above, the general pressure field at any x and y , using the log approximation which we have already shown is valid after a short period of time, becomes,

$$\frac{4\pi kh[p(x,y) - p_1]}{q\mu} = \ln[(C-x)^2 + y^2] - \ln[x^2 + y^2] \quad (1)$$

The pressure gradient along the x axis is,

$$\frac{4\pi kh}{q\mu} \frac{\partial p(x,0)}{\partial x} = -\frac{2}{C-x} - \frac{2}{x} = -\frac{2C}{x(C-x)} = f'(x) \quad (2)$$

where I've also shown Eq. 2 as a general differential, $f'(x)$, as I did before in earlier notes.

From Eq. 9 of the notes titled, Area Sweep Behavior in a Doublet System (p. 13), we know that the area swept at breakthrough is,

$$A_{BT} = -4\pi \int_0^C \frac{dx}{f'(x)} \quad (3)$$

When substituting Eq. 2, and showing the integral to breakthrough, it becomes,

$$A_{BT} = \frac{4\pi}{2C} \int_0^C (C-x) dx \quad (4a)$$

$$= \frac{2\pi}{C} \left[\frac{Cx^2}{2} - \frac{x^3}{3} \right]_0^C = \frac{2\pi}{C} \left(\frac{C^3}{2} - \frac{C^3}{3} \right) = \frac{\pi C^2}{3} \quad (4b)$$

Notice that the constant in Eq. 4b is $\pi/3$, while it was $4\pi/3$ in the earlier areal sweep notes, Eq. 12, p. 14. This is because C was defined to be twice as long here as it was in those earlier notes. The equations are the same.

We also should be interested in calculating the distance moved in the opposite direction, away from the producer along the $-x$ axis. The equation for the pressure gradient is the same as before (Eq. 2), only the integration limits are changed. Further, since the time is the same, we can set this integral equal to the breakthrough integral, from Eq. 4,

$$\int_0^x (C+x)xdx = \int_0^C (C-x)xdx \quad (5a)$$

$$\left| \frac{Cx^2}{2} + \frac{x^3}{3} \right|_0^x = \left| \frac{Cx^2}{2} - \frac{x^3}{3} \right|_0^C \quad (5b)$$

$$\frac{Cx^2}{2} + \frac{x^3}{3} = \frac{C^3}{2} - \frac{C^3}{3} = \frac{C^3}{6} \quad (5c)$$

or

$$\left(\frac{x}{C} \right)^2 \left[3 + 2 \left(\frac{x}{C} \right) \right] = 1 \quad (5d)$$

or

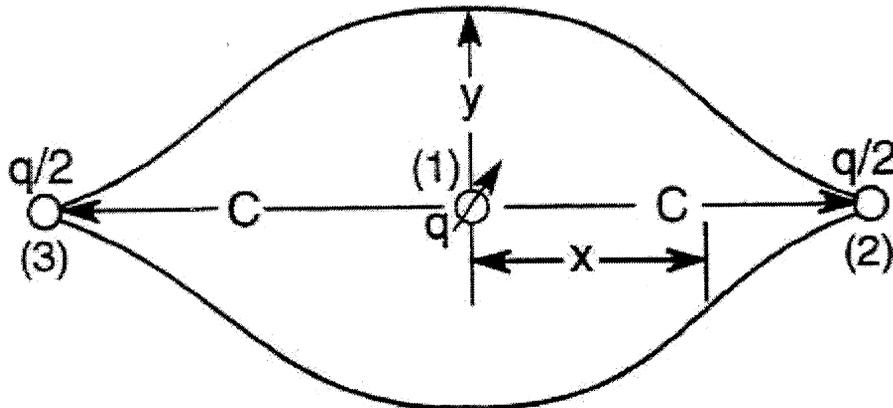
$$\frac{x}{C} = \frac{1}{2} \quad (5e)$$

Thus we have shown that, at breakthrough, the injected fluid moves half as far to the left as it did to the right. This result is the same as we calculated in previous notes, Eq. 53, p. 32.

We will be looking at a number of isolated patterns, and for these patterns we will reverse the injection and producing wells. This idea is important for the more complex patterns, but is not for the doublet system. In the doublet, when the wells are reversed, obviously the solution is unchanged.

4.1.2. The Isolated Inverted Three-Spot

Suppose we have three wells on a line with injection at rate q in the center, and with two producers at rate $q/2$ at distance C along the positive and negative x axis, as shown on the sketch below,



The exponential integral solution to this problem can be written as follows,

$$\frac{4\pi kh}{q\mu} [p(x,y) - p_i] = -Ei\left(-\frac{r_D^2(x,y-1)}{t_D}\right) + Ei\left(-\frac{r_D^2(x,y-2)}{t_D}\right) + \frac{1}{2}Ei\left(-\frac{r_D^2(x,y-3)}{t_D}\right) \quad (6)$$

Here the nomenclature, for example, $r_D(x,y-1)$, is meant to show the radial distance from an x,y point to Well 1, and the same idea holds for the other wells.

As we already know, after a short period of time the exponential integrals approach a log form, as follows,

$$-Ei(-x) = \ln(1/x) - 0.5772 \quad (7a)$$

or, for our case

$$-Ei\left(-\frac{r_D^2}{t_D}\right) = \ln(t_D) - \ln(r_D^2) - 0.5772 \quad (7b)$$

Notice that when Eq. 7b is substituted into Eq. 6, the t_D terms all cancel; there is one $+t_D$ term and two $-t_D/2$ terms. Also the constant, 0.5772, cancels for the same reason. This means that only the log radius terms are needed to define the pressure in the system. We used this idea earlier when developing the doublet equations, but that was only for two wells. The ideas stated here are far more general. It is obvious that, whenever the total production and total injection are equal, the pressure field becomes a simple summation of log radius terms, no matter how many wells are involved. This is a powerful concept that will be used repeatedly in these notes.

We now know, from Eqs. 6 and 7, that we can immediately write the pressure equation for this system. Writing this equation along the x axis between the injector and the right hand producer (Well 2), we get,

$$\frac{4\pi kh [p(x,0) - p_i]}{q\mu} = -\ln(x^2) + \frac{1}{2}\ln(C-x)^2 + \frac{1}{2}\ln(C+x)^2 \quad (8)$$

and when Eq. 8 is differentiated, with respect to x , we get,

$$\frac{4\pi kh}{q\mu} \left(\frac{\partial p}{\partial x}\right) = -\frac{2}{x} - \frac{1}{C-x} + \frac{1}{C+x} \quad (9a)$$

and, after a bit of algebra,

$$\frac{2\pi kh}{q\mu} \left(\frac{\partial p}{\partial x} \right) = \frac{-C^2}{x(C^2 - x^2)} \quad (9b)$$

Again, as we have done before for the doublet, from Eq. 9b after invoking the material balance, we can write an equation for the areal sweep at breakthrough.

$$\begin{aligned} A_{BT} &= \frac{2\pi}{C^2} \int_0^C (C^2 - x^2) x dx \\ &= \frac{2\pi}{C^2} \left[\frac{Cx^2 x^2}{2} - \frac{x^4}{4} \right]_0^C = \frac{2\pi}{C^2} \left(\frac{C^4}{2} - \frac{C^4}{4} \right) \end{aligned} \quad (10a)$$

$$= \frac{2\pi C^2}{4} \quad (10b)$$

Equation 10b can be further simplified by dividing top and bottom by 2, but I'll leave it in this form for reasons that will become apparent later.

From the sketch of this system, it is clear that one streamline will move vertically away from the origin at zero x in the $\pm y$ direction. We will calculate this distance moved by writing the pressure equation for this streamline, as follows,

$$\frac{4\pi kh}{q\mu} [p(0, y) - p_i] = -\ln y^2 + \frac{1}{2} \ln(C^2 + y^2) + \frac{1}{2} \ln(C^2 + y^2) \quad (11)$$

Note that there are two identical terms on the right hand side, because there are two wells of equal strength at equal distances from the vertical centerline. When Eq. 11 is differentiated, with respect to y , it becomes,

$$\frac{4\pi kh}{q\mu} \frac{\partial p}{\partial y} = -\frac{2}{y} + \frac{2y}{C^2 + y^2} = \frac{-2C^2}{y(C^2 + y^2)} \quad (12)$$

Now we can integrate Eq. 12 and compare it with the integral in Eq. 10a to find how far the fluids move in the y direction,

$$\int_0^C (C^2 - x^2) x dx = \int_0^y (C^2 + y^2) y dy \quad (13a)$$

or

$$\frac{C^4}{4} = \frac{C^2 y^2}{2} + \frac{y^4}{4} \quad (13b)$$

or

$$\frac{1}{2} \left(\frac{y}{C} \right)^2 \left[4 + 2 \left(\frac{y}{C} \right)^2 \right] = 1 \quad (13c)$$

or

$$\left(\frac{y}{C} \right)^4 + 2 \left(\frac{y}{C} \right)^2 - 1 = 0 \quad (13d)$$

Equation 13d is a quadratic in $(y/C)^2$. As such, its solution is,

$$\left(\frac{y}{C} \right)^2 = \frac{-2 + \sqrt{4+4}}{2} = \sqrt{2} - 1 \quad (14a)$$

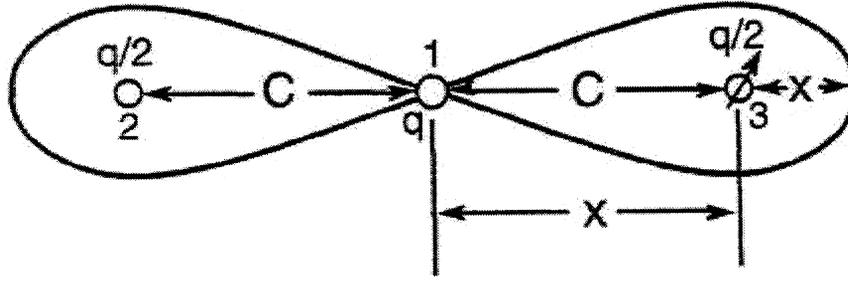
or

$$\left(\frac{y}{C} \right) = \sqrt{\sqrt{2} - 1} = 0.64359 \quad (14b)$$

Thus, Eq. 14b tells us that, at breakthrough, the injected fluid for an inverted isolated three-spot has moved 64% as far in the y direction as it moved in the x direction.

4.1.3. The Normal Isolated Three-Spot

Let us now invert our three-spot pattern to a normal isolated three-spot. There is production at rate q in the center, and injection at rate $q/2$ at two equally spaced wells along the x axis, as indicated on the attached sketch.



Notice, in this system, that the area sweep shapes look like two tear drops centered on the injection wells. I've labeled general distances, x , from the right-hand injector, both toward the producer and away from the producer.

In a manner similar to Eq. 8, we can write the equation for the pressure along the x axis where x is defined as the distance from the right-hand injector toward the producer,

$$\frac{2\pi kh[p(x) - p_i]}{q\mu} = -\frac{1}{2} \ln x - \frac{1}{2} \ln(2C - x) + \ln(C - x) \quad (15)$$

and upon differentiating,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{1}{2x} + \frac{1}{2(2C - x)} - \frac{1}{C - x} \quad (16a)$$

which after some algebra becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = \frac{-C^2}{(2C^2 - 3Cx + x^2)x} \quad (16b)$$

The breakthrough sweep calculation becomes,

$$A_{BT} = \frac{2\pi}{C^2} \int_0^C (2C^2 - 3Cx + x^2) x dx \quad (17a)$$

or,

$$A_{BT} = \frac{2\pi}{C^2} \left[C^2 x^2 - Cx^3 + \frac{x^4}{4} \right]_0^C \quad (17b)$$

$$= \frac{2\pi C^2}{4} \quad (17c)$$

Note that Eq. 17c is identical to Eq. 10b. We should have expected this result, for the pressure fields must be exact mirror images of each other. This behavior will be true for all patterns, so in the notes to follow, only one breakthrough sweep calculation will be made when more complex patterns are addressed.

We can also calculate how far the injected fluid moves beyond Well 2 at breakthrough into Well 1. This can be calculated as before by equating the line integrals, since the times are the same. The pressure equation beyond Well 2 becomes,

$$\frac{2\pi kh[p(x) - p_i]}{q\mu} = -\frac{1}{2}\ln x - \frac{1}{2}\ln(2C+x) + \ln(C+x) \quad (18)$$

which, upon differentiating is,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = -\frac{1}{2x} - \frac{1}{2(2C+x)} + \frac{1}{C+x} \quad (19a)$$

and, when simplified is,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = \frac{-C^2}{(2C^2 + 3Cx + x^2)x} \quad (19b)$$

We can now integrate Eq. 19b and set it equal to the integral in Eqs. 17, with the following result,

$$\frac{2\pi C^2}{4} = \frac{2\pi}{C^2} \int_0^x (2C^2x + 3Cx^2 + x^3) dx \quad (20a)$$

or,

$$\frac{C^4}{4} = C^2x^2 + Cx^3 + \frac{x^4}{4} \quad (20b)$$

or,

$$\left(\frac{x}{C}\right)^4 + 4\left(\frac{x}{C}\right)^3 + \left(\frac{x}{C}\right)^2 = 1 \quad (20c)$$

or,

$$\left(\frac{x}{C}\right)^2 \left[\left(\frac{x}{C}\right)^2 + 4\left(\frac{x}{C}\right) + 4 \right] = 1 \quad (20d)$$

Equation 20d is a perfect square, so its square root can be taken with the result,

$$\left(\frac{x}{C}\right) \left(\frac{x}{C} + 2\right) = 1 \quad (21a)$$

or,

$$\left(\frac{x}{C}\right)^2 + 2\left(\frac{x}{C}\right) - 1 = 0 \quad (21b)$$

Equation 21b is a quadratic in x/C . Its solution is,

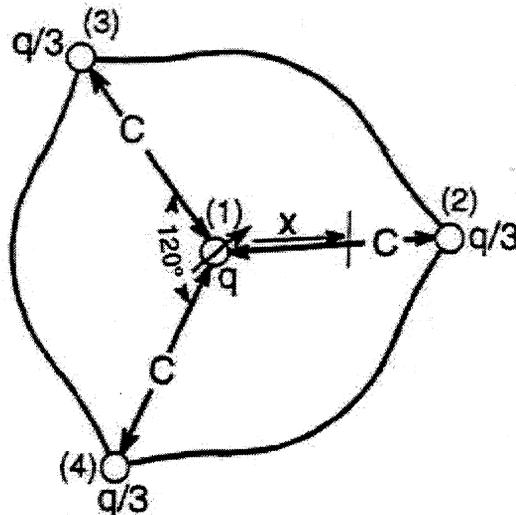
$$\frac{x}{C} = \frac{-2 + \sqrt{4 + 4}}{2} = \sqrt{2} - 1 \quad (22a)$$

$$= 0.41421 \quad (22b)$$

Notice that Eq. 22a, which gives the distance moved away from the outer well in this three well pattern, has numbers similar to Eq. 14a, which was for the inverted pattern. The final constant is not the same, of course, but the $\sqrt{2} - 1$ term is in both expressions. It will be interesting to see if some structural similarities of this sort will be seen in the more complex patterns, which we will address next.

4.1.4 The Isolated Inverted Four-Spot

If we have one injector at the center at q and three producing wells at rates $q/3$ equally spaced at distance C , around the injector, a sketch of the system would look as follows.



Notice in this system, that the angles between the wells are 120° . So the distances along the x axis are now a bit more complex to calculate, for they involve the cosines of the angles.

We can write the general pressure equation along the x axis in radius nomenclature, and then show how to evaluate these radii. The equation is,

$$\frac{4\pi kh[p(x) - p_i]}{q\mu} = -\ln[r^2(1-x)] + \frac{1}{3}\ln[r^2(2-x)] + \frac{1}{3}\ln[r^2(3-x)] + \frac{1}{3}\ln[r^2(4-x)] \quad (23)$$

The nomenclature in Eq. 23 is as follows. For example, the term, $[r^2(4-x)]$, is meant to indicate the radial distance squared between Well 4 and a general point x that lies on the direct streamline between the injection well (Well 1) and Well 2. Similarly for the other radii. The radii from Wells 1 and 2 are obvious and require no further comment, but calculating the radii from Wells 3 and 4 is a bit more complex. From simple trigonometry, these radii are,

$$\begin{aligned} r^2(4-x) &= r^2(3-x) = C^2 + x^2 - 2Cx \cos(120^\circ) \\ &= C^2 + x^2 + Cx \end{aligned} \quad (24)$$

So we can substitute Eq. 24 and the other definitives for the radial distances into Eq. 23, to get,

$$\frac{4\pi kh[p(x) - p_i]}{q\mu} = -\ln x^2 + \frac{1}{3}\ln(C-x)^2 + \frac{2}{3}\ln(C^2 + x^2 + Cx) \quad (25)$$

Note in Eq. 25 that I have combined Wells 3 and 4 into a single term with the constant $2/3$ in front of the logarithm.

Equation 25 can now be differentiated to get,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{1}{x} - \frac{1}{3(C-x)} + \frac{2x+C}{3(C^2 + x^2 + Cx)} \quad (26a)$$

which simplifies to,

$$= \frac{-C^3}{x(C^3 - x^3)} \quad (26b)$$

By analogy with Eq. 10a, p. 68, the areal sweep at breakthrough will be,

$$\begin{aligned} A_{BT} &= \frac{2\pi}{C^3} \int_0^C (C^3 - x^3) x dx \\ &= \frac{2\pi}{C^3} \left[\frac{C^2 x^2}{2} - \frac{x^5}{5} \right]_0^C = \frac{2\pi}{C^3} \left(\frac{C^5}{2} - \frac{C^5}{5} \right) \end{aligned} \quad (27a)$$

$$= \frac{2\pi}{C^3} \left(\frac{3C^5}{10} \right) = \frac{3\pi C^2}{5} \quad (27b)$$

Looking at the sketch above, from the symmetry of the system, it is clear the one streamline will move horizontally to the left of the injector along the $-x$ axis. Two other similar straight streamlines will move at $\pm 60^\circ$ away from the injector; however, the one along the minus x axis is easier to calculate, so we will concentrate on it.

To define this streamline it will be necessary to define its distance from Wells 3 and 4. The angle involved is 60° , so the equation is,

$$r^2(4-x) = r^2(3-x) = C^2 + x^2 - 2Cx \cos 60^\circ = C^2 + x^2 - Cx \quad (28)$$

where the terms, $r^2(4-x)$ and $r^2(3-x)$ mean the distance from Wells 3 and 4 to the streamline along the $-x$ axis.

The equation for the pressure along this $-x$ axis then becomes,

$$\frac{4\pi kh}{q\mu} [p(x,0) - p_i] = -\ln x^2 + \frac{1}{3} \ln(C+x)^2 + \frac{2}{3} \ln[C^2 + x^2 - Cx] \quad (29)$$

When Eq. 29 is differentiated, it becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = -\frac{1}{x} + \frac{1}{3(C+x)} + \frac{2x-C}{3(C^2 + x^2 - Cx)} \quad (30a)$$

which, after considerable algebra, simplifies to,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = \frac{-C^3}{x(C^3 + x^3)} \quad (30b)$$

We can now use Eq. 30b to set up a line integral and set it equal to the integral in Eq. 27 since we are looking at these integrals at the same time. The result is,

$$\frac{3\pi C^2}{5} = \frac{2\pi}{C^3} \int_0^x (C^3 + x^3) x dx \quad (31a)$$

or

$$\frac{3C^2}{5} = \frac{2}{C^3} \left(\frac{C^3 x^2}{2} + \frac{x^5}{5} \right) \quad (31b)$$

or

$$\frac{1}{3} \left(\frac{x}{C} \right)^2 \left[5 + 2 \left(\frac{x}{C} \right)^3 \right] = 1 \quad (31c)$$

or

$$2\left(\frac{x}{C}\right)^5 + 5\left(\frac{x}{C}\right)^2 - 3 = 0 \quad (31d)$$

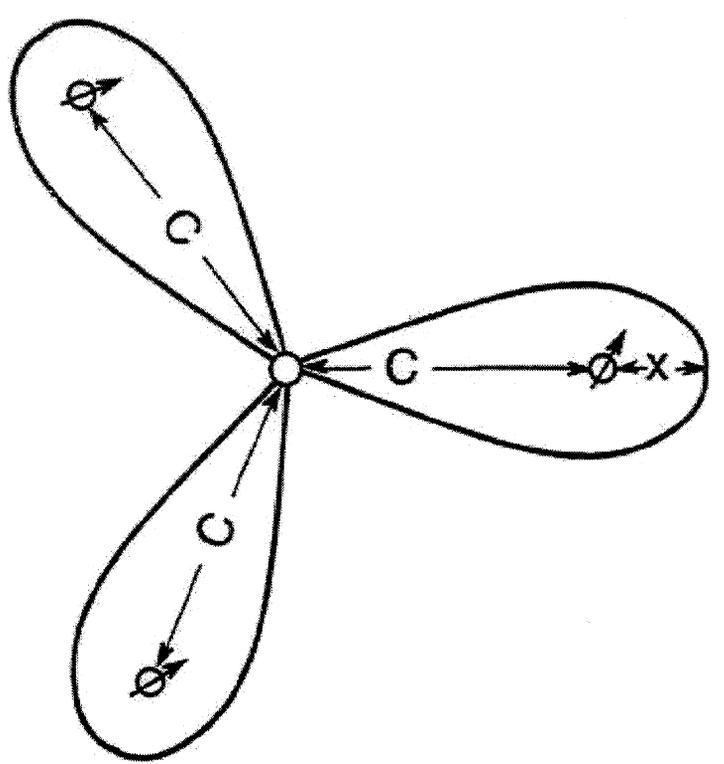
Equation 31d cannot be solved analytically, so it must be solved by trial and error. Its solution is,

$$\left(\frac{x}{C}\right)_{BT} = -0.72212 \quad (32)$$

The minus sign was inserted into the equation to remind us it indicates movement to the left. As the equation was written, the solution to Eq. 31d is + 0.72212. Notice that, at breakthrough, the injected fluid has moved roughly $\frac{3}{4}$ as far to the left as it did to the right. In fact it has moved considerably beyond the vertical straight line between Wells 3 and 4. Their distances are only 0.5C to the left.

4.1.5. The Isolated Normal Four-Spot

The regular isolated four-spot will have three injectors evenly spaced at rates $q/3$ and a center producer at rate q as indicated in the following sketch.



As we've seen in the other patterns, the area swept at breakthrough will be the same for this pattern as it was for the inverted four-spot. So I will only address the question of movement in the tear-drop shape beyond the injection wells, as indicated in the figure on the right-hand well.

By now it should be clear how to formulate this problem. The effects of all the wells are added, as follows,

$$\frac{4\pi kh}{q\mu} [p(x) - p_i] = -\frac{1}{3} \ln x^2 - \frac{2}{3} \ln [C^2 + (C+x)^2 + C(C+x)] + \ln(C+x)^2 \quad (33a)$$

or,

$$\frac{2\pi kh}{q\mu} [p(x) - p_i] = -\frac{1}{3} \ln x - \frac{1}{3} \ln(3C^2 + 3Cx + x^2) + \ln(C+x) \quad (33b)$$

When Eq. 33b is differentiated, it becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{1}{3x} - \frac{3C+2x}{3(3C^2 + 3Cx + x^2)} + \frac{1}{C+x} \quad (34a)$$

$$= \frac{-C^3}{x(3C^3 + 6C^2x + 4Cx^2 + x^3)} \quad (34b)$$

As before we can integrate Eq. 34b over x and set it equal to the integral in Eq. 27, with the following result,

$$\frac{3\pi C^2}{5} = \frac{2\pi}{C^3} \int_0^x (3C^3 + 6C^2x + 4Cx^2 + x^3) x dx \quad (35a)$$

or,

$$\frac{3C^2}{5} = \frac{2}{C^3} \left(\frac{3C^3 x^2}{2} + 2C^2 x^3 + Cx^4 + \frac{x^5}{5} \right) \quad (35b)$$

$$\text{or,} \quad \frac{1}{3} \left(\frac{x}{C} \right)^2 \left[2 \left(\frac{x}{C} \right)^3 + 10 \left(\frac{x}{C} \right)^2 + 20 \left(\frac{x}{C} \right) + 15 \right] = 1 \quad (35c)$$

Equation 35d must be solved by trial and error for x/C . Its solution is,

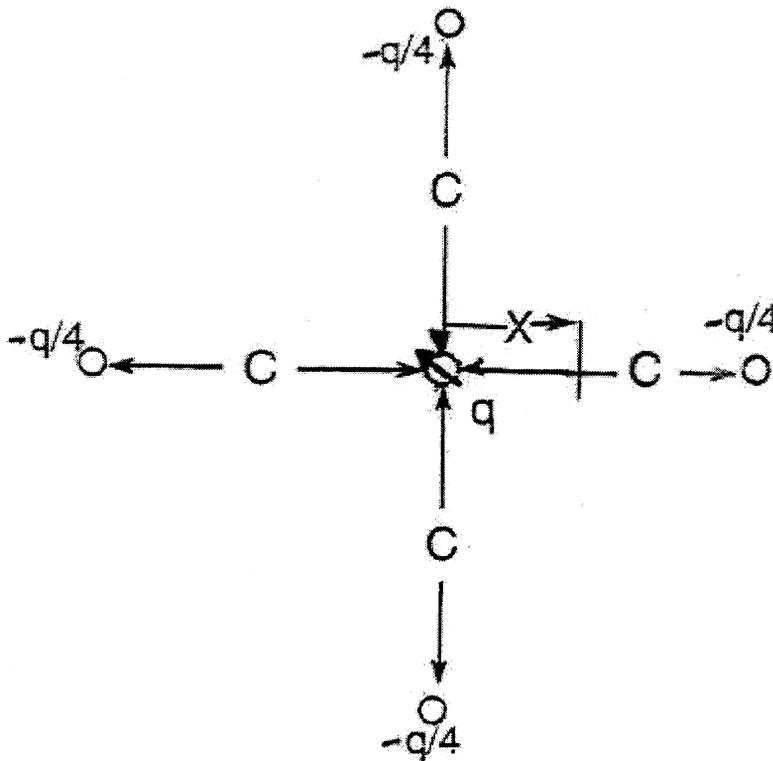
$$\frac{x}{C} = 0.35721 \quad (36)$$

We can compare Eq. 35c for the regular pattern, with Eq. 31c for the inverted pattern. I had hoped some simple relationship could be found between them as we found for the three-spot. Unfortunately, I did not find one.

4.1.6. Isolated Inverted Five-Spot

Let us look at an isolated inverted five-spot pattern with rate, q at a central injector and rate, $-q/4$ at each of four producers, as shown on the sketch below.

It is clear from this sketch, that the two wells above and below the injector are exactly the same distance from the breakthrough streamline directly along the x axis. As a result, the log approximation to the pressure field along the x axis, between the injector and the right-hand producer, is as follows,



$$\frac{4\pi kh}{q\mu} [p(x) - p_i] = -\ln x^2 + \frac{1}{4} \log(C+x)^2 + \frac{1}{4} \ln(C-x)^2 + \frac{1}{2} \ln(C^2 + x^2) \quad (37)$$

Notice in Eq. 37, that the two producing wells above and below the injector are shown together by using $\frac{1}{2}$ in front of the log term rather than $\frac{1}{4}$ as was used for the single right and left producers.

To calculate the pressure gradient, Eq. 37 can be differentiated, with the following result,

$$\frac{4\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{2}{x} + \frac{1}{2(C+x)} - \frac{1}{2(C-x)} + \frac{x}{C^2 + x^2} \quad (38a)$$

which simplifies to,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = \frac{-C^4}{x(C^4 - x^4)} \quad (38b)$$

By analogy with Eq. 10a, p. 68, as we have done before, the areal sweep at breakthrough will be,

$$\begin{aligned} A_{BT} &= \frac{2\pi}{C^4} \int_0^C (C^4 - x^4) x dx \\ &= \frac{2\pi}{C^4} \left[\frac{C^4 x^2}{2} - \frac{x^6}{6} \right]_0^C = \frac{2\pi}{C^4} \left(\frac{C^6}{2} - \frac{C^6}{6} \right) \end{aligned} \quad (39a)$$

$$= \frac{2\pi}{C^4} \left(\frac{2C^6}{6} \right) = \frac{4\pi C^2}{6} \quad (39b)$$

Obviously, the fraction in Eq. 39b can be further simplified from $\frac{4}{6}$ to $\frac{2}{3}$, but I will leave it in this form for reasons that will become clear next.

4.1.7. Recurrence Relations for Breakthrough Sweep

Let us look at all the breakthrough integrals we have solved so far--from the doublet to the five-spot. The table below summarizes these calculations.

Breakthrough Integrals for Isolated Patterns

Pattern	Breakthrough Integral	Areal Sweep At Breakthrough, A_{BT}
2 Spot	$\frac{2\pi}{C} \int_0^C (C - x) x dx$	$\frac{\pi C^2}{3}$
3 Spot	$\frac{2\pi}{C^2} \int_0^C (C^2 - x^2) x dx$	$\frac{2\pi C^2}{4}$
4 Spot	$\frac{2\pi}{C^3} \int_0^C (C^3 - x^3) x dx$	$\frac{3\pi C^2}{5}$
5 Spot	$\frac{2\pi}{C^4} \int_0^C (C^4 - x^4) x dx$	$\frac{4\pi C^2}{6}$

There is a remarkable family resemblance between all these results. The integrals were all derived from the superposition of the effects of the wells; and when there are a large number of wells, the pressure equation becomes quite lengthy. See Eq. 37, for example. However, when the equation is differentiated it always reduces into the simple functions shown in the table. For example, notice how Eq. 38a simplifies to Eq. 38b.

Notice the logical sequence in the table of integrals. From this table, one would presume that the integral for, for example, the isolated seven-spot would be,

$$A_{BT}(7\text{-spot}) = \frac{6\pi C^2}{8} \quad (40b)$$

and if we were to work out the problem, we would find that we were right.

Another important idea can be gleaned from the results in the table above. We can write a general equation for the areal sweep in terms of the number of wells, n , as follows

$$(A_{BT})_n = \frac{(n-1)\pi C^2}{(n+1)} \quad (41)$$

Notice what happens to Eq. 41 as the number of wells increases toward infinity. The result is that it degenerates to the area of the circle on which the outer wells lie. We might have anticipated this, but it is interesting that the breakthrough "peaks" that we see when only a few wells are present, gradually disappear as more wells are added. Or conversely, the "tear drops" we see in a regular pattern, with few wells, no longer move outside the pattern as the number of wells approaches infinity.

4.1.8. Distance Away from Injector

We should also be interested in the linear distances moved away from the injection well at the time of breakthrough. Looking again at the sketch of the inverted five-spot pattern and considering the symmetry of the system, it should be clear that the only streamlines that will be straight from the injector are those that lie 45° , 135° , 225° and 315° around the injector. Actually, it may be easier to visualize these, and the corresponding streamlines in all the patterns, by recognizing that these streamlines are always exactly halfway between adjacent producers. This makes good sense from a consideration of the pressure and flow fields that must develop. These streamlines divide the areas of influence of the producing wells, and they extend as straight lines out to infinity.

It would be a bit tedious to calculate the straight line distance for the inverted five-spot, for the line integral is not along a convenient coordinate. Instead, let us see if we can glean some information from the integrals for the previous patterns, as summarized in the table below.

In the second column, this table shows the form of the integrals to breakthrough for the first three patterns. Notice that these integrals all have the same general format, with the power functions increasing with each increase in the number of wells. Using the obvious recursion relationship, the breakthrough integral for the inverted 5-spot would be expected to be as follows,

$$\frac{4\pi C^2}{6} = \frac{2\pi}{C^4} \int_0^z (C^4 + z^4) z dz \quad (42)$$

Is Eq. 42 correct? I've not proven it is; but I'll wager it is--at very high odds. We could integrate Eq. 42 to show a calculation for the distance moved, z . But this, too, seems unnecessary. By analogy with the equations in the third column in the table below, we would expect the equation for the distance moved to be,

$$\frac{1}{4} \left(\frac{z}{C} \right)^2 \left[6 + 2 \left(\frac{z}{C} \right)^4 \right] = 1 \quad (43)$$

Again I've not proven Eq. 43 is correct; but all logic says it must be.

**Integrals Defining Distance Moved in Inverted Patterns
Along Linear Streamlines Away From Injection**

System	Straight Line Integral away from Injector	Equation for Distances Moved
2-Spot (Doublet)	Eq. 5a $\frac{2\pi}{C} \int_0^x (C+x) x dx$	Eq. 5d $\left(\frac{x}{C} \right)^2 \left[3 + 2 \left(\frac{x}{C} \right) \right] = 1$
Inverted 3-Spot	Eq. 13a $\frac{2\pi}{C^2} \int_0^y (C+y^2) y dy$	Eq. 13c $\frac{1}{2} \left(\frac{y}{C} \right)^2 \left[4 + 2 \left(\frac{y}{C} \right)^2 \right] = 1$
Inverted 4-Spot	Eq. 31a $\frac{2\pi}{C^3} \int_0^x (C^3 + x^3) x dx$	Eq. 31c $\frac{1}{3} \left(\frac{x}{C} \right)^2 \left[5 + 2 \left(\frac{x}{C} \right)^3 \right] = 1$

Based on these results we can look at the general form of the distance equations along the linear dividing streamline at breakthrough of any inverted pattern. By analogy with Eqs. 5d, 13c, 31c and 43, the equation should be,

$$\left(\frac{1}{n-1}\right)\left(\frac{z}{C}\right)^2 \left[(n+1) + 2\left(\frac{z}{C}\right)^{n-1} \right] = 1 \quad (44a)$$

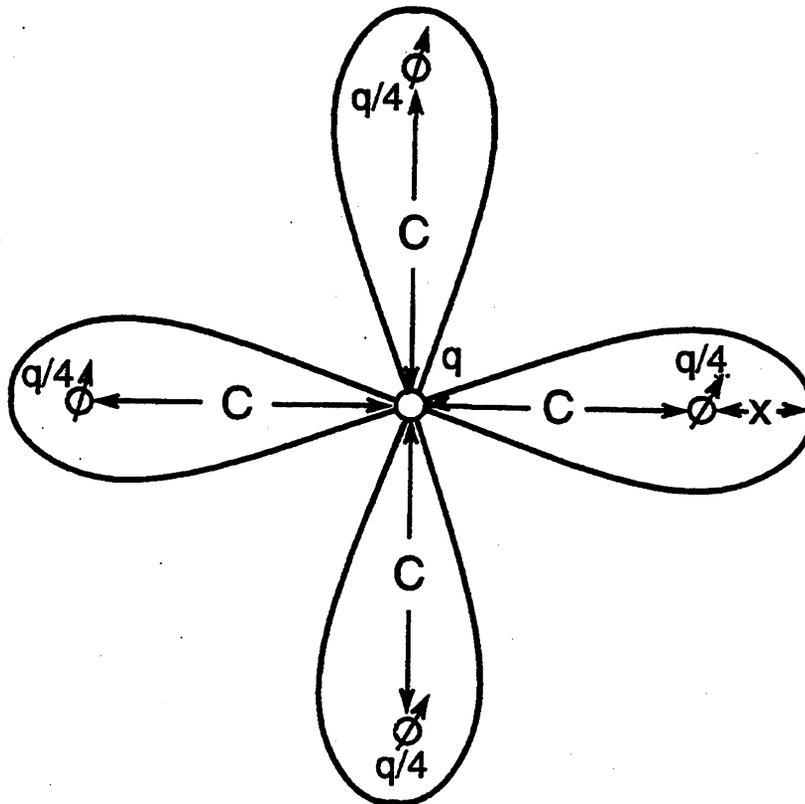
or

$$\frac{(n+1)}{(n-1)}\left(\frac{z}{C}\right)^2 + \left(\frac{2}{n-1}\right)\left(\frac{z}{C}\right)^{n+1} = 1 \quad (44b)$$

As n approaches infinity, it is clear from Eq. 44b that z/C must approach 1.0. Thus the behavior of these streamlines again is logical from our knowledge of the system. The system becomes a circle and the flow within it is purely radial.

4.1.9. Isolated Normal Five-Spot

The geometry of an isolated normal five-spot pattern, with rate q at the center producer and rates $q/4$ at the exterior injectors, is shown below in a sketch which includes the breakthrough pattern.



Of course the breakthrough sweep efficiency for this pattern will be exactly the same as for the inverted pattern, so the only term we need to calculate is the distance moved along the "teardrop" at the time of breakthrough into the center well. This distance is labeled x to the right on the figure.

The general pressure field to the right of the right-hand injector is defined as follows.

$$\frac{4\pi kh}{q\mu} [p(x) - p_i] = \ln(C+x)^2 - \frac{1}{4} \ln x^2 - \frac{1}{4} \ln(2C+x)^2 - \frac{1}{2} \ln [C^2 + (C+x)^2] \quad (45)$$

Notice that the last term in Eq. 45 is multiplied by $\frac{1}{2}$ rather than $\frac{1}{4}$ because there are two wells at this distance.

When Eq. 45 is differentiated with respect to x , it becomes,

$$\frac{4\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = \frac{2}{C+x} - \frac{1}{2x} - \frac{1}{2(2C+x)} - \frac{C+x}{2C^2 + 2Cx + x^2} \quad (46a)$$

which, after considerable algebra, simplifies to,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = \frac{-C^4}{x(C+x)(2C+x)(2C^2 + 2Cx + x^2)} \quad (46b)$$

As we have done before, we can set the time for the movement along the streamline of Eq. 46b equal to the time for breakthrough. The resulting equation will be,

$$\frac{4\pi C^2}{6} = \frac{2\pi}{C^4} \int_0^x (2C^2 + 2Cx + x^2)(2C+x)(C+x) dx \quad (47a)$$

or,

$$= \frac{2\pi}{C^4} \int_0^x (4C^4x + 10C^3x^2 + 10C^2x^3 + 5Cx^4 + x^5) dx \quad (47b)$$

or, after integration and some algebra,

$$\frac{1}{2} \left(\frac{x}{C} \right)^2 \left[\left(\frac{x}{C} \right)^4 + 6 \left(\frac{x}{C} \right)^3 + 15 \left(\frac{x}{C} \right)^2 + 20 \left(\frac{x}{C} \right) + 12 \right] = 1 \quad (47c)$$

Equation 47c must be solved by trial and error for x/C . Its solution is,

$$\frac{x}{C} = 0.31607 \quad (48)$$

4.1.10 Distance Moved in Normal Patterns

Recalling that the equations for area and distance for the inverted patterns showed a logical sequence, it seems likely that the equations similar to Eqs. 45-48 would also reveal some symmetry for the various patterns. If so, they could be extended to a greater number of wells. To test this possibility, the equations and results from all the normal isolated patterns tested thus far are listed in the table below.

At first glance, no recurrence relation jumps out at the casual observer, but upon more careful reflection, there *is* some logical progression in these integrals. The first integral for the two-spot contains two terms which define the distances the line integral location is from each well. These distances are multiplied together in the integral.

Line Integral for Movement in Opposite Direction -- Normal Isolated Pattern

Isolated Pattern	Integral for Opposite Streamline	x/C at Breakthrough
Doublet 2-Spot	$\int_0^x (C+x)x dx$	0.50000
Normal 3-Spot	$\int_0^x (2C+x)(C+x)x dx$	0.41421
Normal 4-Spot	$\int_0^x (3C^2 + 3Cx + x^2)(C+x)x dx$	0.35721
Normal 5-Spot	$\int_0^x (2C^2 + 2Cx + x^2)(2C+x)(C+x)x dx$	0.31607

The second integral is for the three-spot pattern. It contains three terms multiplied together, and each term is the distance from the line integral location, x , to each well. It is interesting that we get this result. Logically we might have expected the center well to be counted twice (that is, squared), because it is at double the rate of the other wells.

The normal four-spot does not appear to be in any logical sequence at first glance. Particularly notice that there are only three terms, while there are four wells. However, consider the term, $3C^2 + 3Cx + x^2$. This is the square of the distance to the wells that are at angles of $\pm 120^\circ$ from the streamline of interest. There are two wells, and they are

equally spaced, so the effect is to multiply their distances. So again, the distances from each well are multiplied together inside the integral.

Finally, we look at the five-spot. We see that there are three terms which correspond to the three wells which lie along the x axis; the right-hand injector, the middle producer and the left-hand injector. The additional term, $2C^2 + 2Cx + x^2$ is the square of the distance to each well on the y axis above and below the producer. So again, in this integral, the distances from each well are multiplied together to form the integral which defines the distance moved away from the injectors.

These concepts can be generalized for any number of wells in a balanced pattern. We can write an equation similar to Eq. 47a to define the distance moved away from any normal n -spot, as follows,

$$\frac{(n-1)\pi C^2}{(n+1)} = \frac{2\pi}{C^{n-1}} \int_0^x \prod_1^n [r(n-x)] dx \quad (49)$$

where the term $r(n-x)$ is meant to indicate the radial distance from well n to the general location, x , along the line integral. The symbol \prod , is commonly used to indicate that all the terms in the series are multiplied together. There is no convenient way to simplify the integral of Eq. 49 further to show how it behaves as more wells are added; however, another approach shows some merit, as indicated next.

As formulated in Eq. 49, each pattern requires extensive multiplication and then integration. However, there is another way to formulate these distance equations. Suppose that, instead of putting the variables in terms of x , the distance beyond the external injectors, we were to define the variable from the center of the system. We will call this variable z , for convenience. Then we define,

$$z = C + x \quad (50a)$$

and,

$$dz = dx \quad (50b)$$

The limits of integration then will become,

$$x = 0 \quad z = C \quad (50c)$$

$$x = x \quad z = C + x \quad (50d)$$

When these definitions are substituted into the breakthrough equations for the two-spot (Eq. 5a), the normal three-spot (Eq. 20a), the normal four-spot (Eq. 35a) and the normal five-spot (Eq. 47a), the table on the following page results.

The results in this table clarify the nature of the behavior of the flow from the injectors in the direction opposite to the producers, and also it clarifies the behavior of the patterns as more wells are added. The general equation now becomes simply,

$$\frac{(n-1)\pi C^2}{(n+1)} = \frac{2\pi}{C^{n-1}} \int_C^{C+x} (z^{n-1} - C^{n-1}) z dz \quad (51)$$

The general integral format can be easily evaluated at any finite n , and also as n approaches infinity. It is clear that, for the integral to be bounded as n approaches infinity, the limit $C+x$ must approach C . This result is as we should have anticipated.

**Line Integral for Movement in Opposite
Direction - Normal Isolated Patterns Where $z = C + x$**

Isolated Pattern	Integral for Opposite Streamline
Doublet (2-Spot)	$\frac{2\pi}{C} \int_C^{C+x} (z - C) z dz = \frac{\pi C^2}{3}$
Normal (3-Spot)	$\frac{2\pi}{C^2} \int_C^{C+x} (z^2 - C^2) z dz = \frac{2\pi C^2}{4}$
Normal (4-Spot)	$\frac{2\pi}{C^3} \int_C^{C+x} (z^3 - C^3) z dz = \frac{3\pi C^2}{5}$
Normal (5-Spot)	$\frac{2\pi}{C^4} \int_C^{C+x} (z^4 - C^4) z dz = \frac{4\pi C^2}{6}$ where $z = C + x$

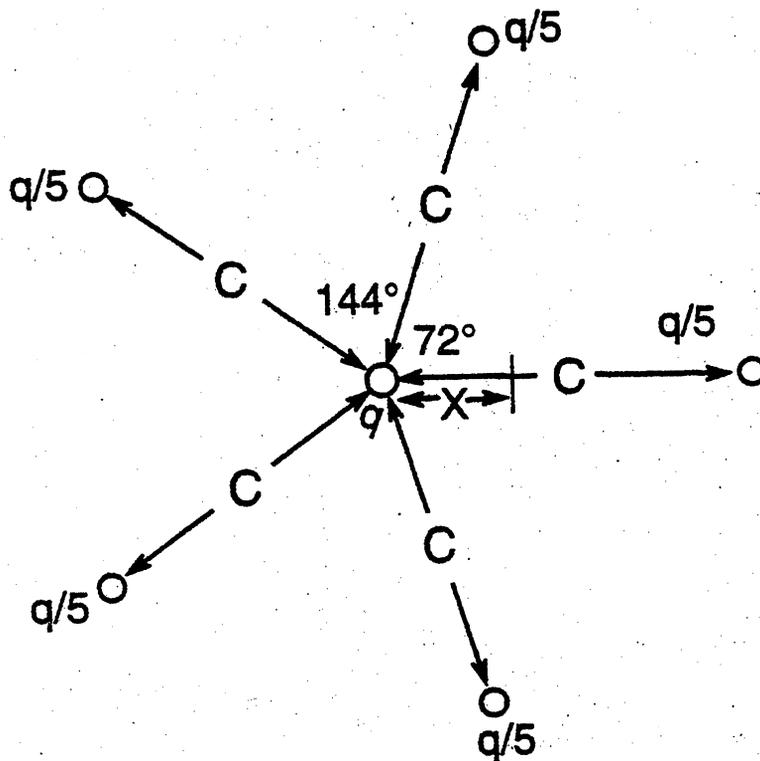
Although these integrals are easy to perform, their evaluation still requires a trial and error procedure. However, since the integrals are fairly simple in form, it appears they would be easier to evaluate than Eq. 47 was, for example.

4.1.11 Isolated Inverted Six-Spot Pattern

Since we have generalized the behavior of the patterns from the results of the two-spot through the five-spot, it doesn't seem necessary to look at any other patterns. But one concept may be a bit worrisome. Remember that the

distances involved included the general length of the side of a triangle, which includes the cosine of the angle in its formulation. For example, look at Eq. 24 of these notes, p. 73. So far, all the angles used have had simple algebraic forms for their cosines, but in general, this is not true. The six-spot is a case in point.

We will not calculate all the six-spot terms, but we will see whether the log summation terms simplify for this system the way they did for all the other patterns. Let's first picture the inverted isolated six-spot.



By analogy with the patterns we have looked at before (for example, consider Eq. 24, p. 73), we can write the general pressure equation for this system as follows,

$$\frac{4\pi kh[p(x) - p_i]}{q\mu} = -\ln x^2 + \frac{1}{5} \ln(C-x)^2 + \frac{2}{5} \ln[C^2 + x^2 - 2Cx \cos(72^\circ)] + \frac{2}{5} \ln[C^2 + x^2 - 2Cx \cos(144^\circ)] \quad (52)$$

Notice the $\frac{2}{5}$ multiplier on the terms that relate to two wells. When Eq. 52 is differentiated, it becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{1}{x} - \frac{1}{5C-x} + \frac{(2x-2C\cos 72^\circ)}{5(C^2+x^2-2Cx\cos 72^\circ)} + \frac{(2x-2C\cos 144^\circ)}{5(C^2+x^2-2Cx\cos 144^\circ)} \quad (53)$$

Equation 53 is so complex that, from now on I'll discuss its numerator and denominator separately. Its numerator is,

$$\begin{aligned} \text{Eq. 53 numerator} &= -5(C-x)(C^2+x^2-2Cx\cos 72^\circ)(C^2+x^2-2Cx\cos 144^\circ) \\ &\quad -x(C^2+x^2-2Cx\cos 72^\circ)(C^2+x^2-2Cx\cos 144^\circ) \\ &\quad +x(C-x)(2x-2C\cos 72^\circ)(C^2+x^2-2Cx\cos 144^\circ) \\ &\quad +x(C-x)(2x-2C\cos 144^\circ)(C^2+x^2-2Cx\cos 72^\circ) \end{aligned} \quad (54a)$$

After considerable multiplication and combination of terms, Eq. 54a simplifies to,

$$\begin{aligned} \text{Eq. 53 numerator} &= -5C^2 + 4C^4x - 6C^3x^2 + 4C^2x^3 - Cx^4 \\ &\quad -(-8C^4x + 6C^3x^2 - 4C^2x^3 + 2Cx^4)(\cos 144^\circ + \cos 72^\circ) \\ &\quad -x(12C^3x^2 - 8C^2x^3)(\cos 72^\circ \cos 144^\circ) \end{aligned} \quad (54b)$$

It's clear that if we are to evaluate Eq. 54b, we need some trigonometric relationships between $\cos 72^\circ$ and $\cos 144^\circ$.

One such useful relationship is,

$$\cos 144^\circ = 2\cos^2 72^\circ - 1 \quad (55a)$$

Thus,

$$\cos 144^\circ + \cos 72^\circ = \cos 72^\circ + 2\cos^2 72^\circ - 1 \quad (55b)$$

Another useful relationship is,

$$\cos 144^\circ \cos 72^\circ = \frac{\cos(144^\circ - 72^\circ)}{2} + \frac{\cos(144^\circ + 72^\circ)}{2} \quad (56a)$$

$$= \frac{\cos(72^\circ)}{2} + \frac{\cos(144^\circ)}{2} \quad (56b)$$

$$= \frac{\cos(72^\circ)}{2} + \cos^2(72^\circ) - \frac{1}{2} \quad (56c)$$

Note that in Eq. 56b, I've used the fact that $\cos(216^\circ) = \cos(144^\circ)$. When Eqs. 55b and 56c are substituted into Eq. 54b, it simplifies to,

$$\text{Eq. 53 numerator} = -5C^5 - (4C^4x - 6C^3x^2 + 4C^2x^3 - Cx^4)(1 - 2\cos 72^\circ - 4\cos^2 72^\circ) \quad (54c)$$

But it is possible to prove that,

$$1 - 2\cos 72^\circ - 4\cos^2 72^\circ = 0 \quad (57)$$

Equation 57 is not generally true for all angles, but it is true for 72° , that is, $2\pi/5$. Thus Eq. 54c simplifies to,

$$\text{Eq. 53 numerator} = -5C^5 \quad (54d)$$

just as we would have predicted.

The denominator for Eq. 53 is,

$$\text{Eq. 53 denominator} / 5x = (C - x)(C^2 + x^2 - 2Cx\cos 72^\circ)(C^2 + x^2 - 2Cx\cos 144^\circ) \quad (58a)$$

After considerable multiplication and simplification, Eq. 58a becomes,

$$\begin{aligned} \text{Eq. 53 denominator} / 5x &= C^5 - x^5 - C^4x + 2C^3x^2 - 2C^2x^3 + Cx^4 \\ &\quad - (\cos 72^\circ + \cos 144^\circ)(2C^4x - 2C^3x^2 + 2C^2x^3 - 2Cx^4) \\ &\quad + (\cos 72^\circ \cos 144^\circ)(4C^3x^2 - 4C^2x^3) \end{aligned} \quad (58b)$$

Again we can substitute Eqs. 55b and 56c into Eq. 58b, and after considerable simplification and combining of terms, we get,

$$\begin{aligned} \text{Eq. 53 denominator} / 5x &= C^5 - x^5 + C^4x - 2C^3x^2 + 2C^2x^3 - Cx^4 \\ &\quad + (-2C^4x + 4C^3x^2 - 4C^2x^3 + 2Cx^4)(\cos 72^\circ) \\ &\quad + (-4C^4x + 8C^3x^2 - 8C^2x^3 + 4Cx^4)(\cos^2 72^\circ) \end{aligned} \quad (58c)$$

Again we can invoke Eq. 57 to eliminate most of this equation, with the result,

$$\text{Eq. 53 denominator} / 5x = C^5 - x^5 \quad (58d)$$

or

$$\text{Eq. 53 denominator} = (C^5 - x^5)(5x) \quad (58e)$$

Thus we can see that the equation for the pressure gradient, Eq. 53, has reduced to the expression we expected, vis.,

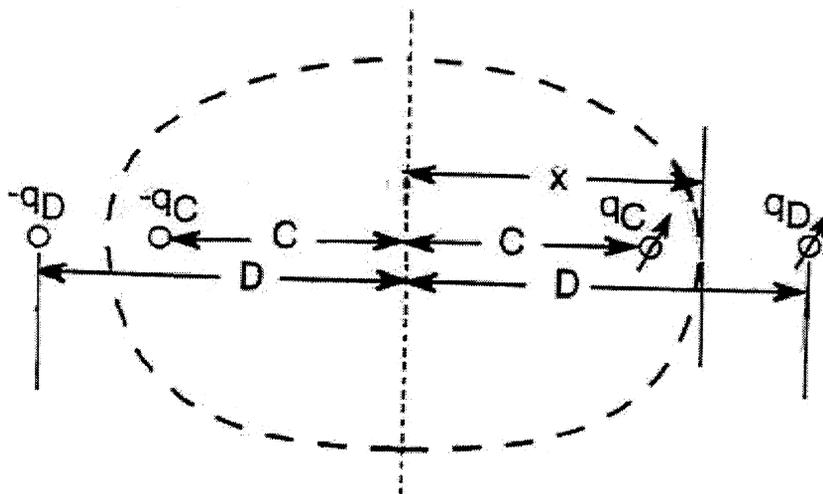
$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = \frac{-5C^5}{5x(C^5 - x^5)} = \frac{-C^5}{(C^5 - x^5)x} \quad (53b)$$

To be complete, I probably ought to show the distances moved along the straight streamlines for both the inverted and normal isolated six-spot patterns, as we did for the other patterns. It is clear from the previous pages that the algebra would be quite tedious; and it also seems logical that the recurrence relations we have anticipated are very likely to be correct. That is, we should expect that the resulting cosign terms will cancel as they did in Eqs. 54 and 58 due to the identity of Eq. 57. I've not gone through the tedious algebra to prove these statements are correct; but again, I'll wager they are, at very large odds.

I'll turn now to some other patterns that might be of interest to petroleum and groundwater engineers.

4.2. THE DOUBLE DOUBLET

First I should define what I mean by the term "The Double Doublet." Suppose we have four wells on a line: two injectors and two producers. There will be an inner injector/producer pair at one rate and an outer injector/producer pair at another rate. And the spacing will be the same between the pairs of injectors and producers. A sketch of this idea is shown below.



This system has interesting properties. You'll note that I have drawn a "rugby ball" shaped dashed line on the figure. This line is there to indicate that this four well system automatically sets up two regions. All the fluid injected at location C on the right must be produced at the inner well at distance C on the left. Also all the injected fluid at the well at distance D on the right eventually is produced at its corresponding well at distance D on the left. Thus the inner wells carve out an impermeable barrier as far as the outer wells are concerned. This idea was used a few years ago in the San Jose area to help limit salt water intrusion into a fresh water aquifer (Ramey, 1973).

This idea of a "carved out" area is so startling that it may be hard to believe, so I'll try to prove it logically. Suppose my statement was wrong. In that case, some of the fluid from the well at C on the right would have to eventually end up in the well at D on the left. If this occurred, then it would be necessary that some of the fluid from the well at D on the right would be produced at the well at C on the left. But if these two statements were true, then it would be necessary for some of the flow lines to cross each other. And, of course, that is impossible. Thus the carved-out zone must exist, as I've pictured it in the sketch above.

To date, I have not worked out the equation for this carved out zone. One might guess that it is an ellipse, but I don't know. It is relatively easy, however to locate the stagnation point between the wells at the distance x indicated on the diagram. We merely write the general pressure equation and differentiate it. Then we set the derivative equal to zero to define the no-flow distance, x , as a function of the other distance variables and the rates in the wells.

The pressure equation for this system along the x axis, is,

$$\frac{4\pi kh[p(x,0) - p_i]}{\mu} = q_C \ln(C+x)^2 + q_D \ln(D+x)^2 - q_C \ln(x-C)^2 - q_D \ln(D-x)^2 \quad (54)$$

and its derivative is,

$$\frac{2\pi kh}{\mu} \frac{\partial p(x)}{\partial x} = \frac{q_C}{C+x} + \frac{q_D}{D+x} - \frac{q_C}{x-C} + \frac{q_D}{D-x} \quad (55a)$$

and after cross multiplying,

$$\begin{aligned} &= \frac{q_C(D^2 - x^2)(x-C) + q_D(x^2 - C^2)(D-x)}{(x^2 - C^2)(D^2 - x^2)} \\ &\quad - \frac{q_C(D^2 - x^2)(C+x) + q_D(x^2 - C^2)(D+x)}{(x^2 - C^2)(D^2 - x^2)} \end{aligned} \quad (55b)$$

$$\frac{2\pi kh}{\mu} \frac{\partial p(x)}{\partial x} = \frac{2Dq_D(x^2 - C^2) - 2Cq_C(D^2 - x^2)}{(x^2 - C^2)(D^2 - x^2)} \quad (55c)$$

We need to evaluate Eq. 55c, with the gradient zero, to find the stagnation point; so we set the numerator equal to zero, as follows,

$$2Dq_D(x^2 - C^2) = 2Cq_C(D^2 - x^2) \quad (56a)$$

and we can solve for x ,

$$x^2 = \frac{CD(Dq_C + Cq_D)}{(Cq_C + Dq_D)} \quad (56b)$$

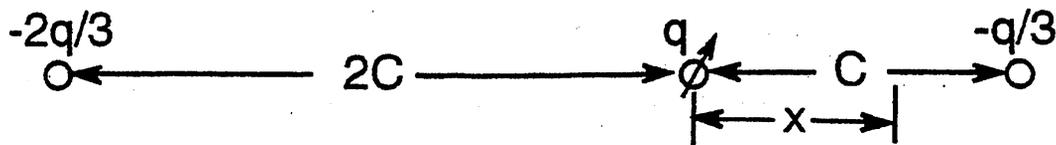
It is of interest to find out how Eq. 56b behaves under various conditions. For example, if the rates in all wells are equal, Eq. 56b degenerates to,

$$x^2 = CD \quad (56c)$$

Thus the stagnation point is at the geometric mean distance to the wells -- not the arithmetic mean, that we might have guessed. When the rate is higher in the outside wells ($q_D > q_C$), the stagnation point moves closer to the inner wells as we might have anticipated; and when the rate is high in the inner wells, the stagnation point moves closer to D .

4.3. UNEQUAL WELL RATES (INVERTED THREE-WELL SYSTEM)

When the rates at the wells all differ from each other, with the total rate still equal to zero, the idea of a steady state system still holds true, for all the terms but the log radii will still cancel. Clearly there is a huge variety of such well arrays that could be addressed. I'll only show two three-well systems to show how the mathematics behaves for such unbalanced patterns. The math is considerably more complex. Consider the inverted pattern sketched below.



For this system, it is clear that the breakthrough streamlines will lie along the x axis at zero y . The pressure equation along the positive x axis is,

$$\frac{4\pi kh[p(x) - p_i]}{q\mu} = \frac{1}{3} \ln(C-x)^2 + \frac{2}{3} \ln(2C+x)^2 - \ln(x)^2 \quad (57)$$

Notice in Eq. 57 that the multipliers on the log terms indicate their individual rates. This concept is always true, and has been true for all the problems addressed in this section. But it was not emphasized before.

To get the flow rates from the injector toward the right-hand producer, we need to differentiate Eq. 57 with respect to x as we have done before. The result is,

$$\frac{4\pi kh}{q\mu} \frac{\partial p}{\partial x} = \frac{2}{3(C-x)} + \frac{4}{3(2C+x)} - \frac{2}{x} \quad (58a)$$

or,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = \frac{-x(2C+x) + 2x(C-x) - 3(C-x)(2C+x)}{3x(C-x)(2C+x)} \quad (58b)$$

which, after considerable simplification becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = \frac{-C(2C-x)}{x(C-x)(2C+x)} = \frac{-C(2C-x)}{(-x^3 - Cx^2 + 2C^2x)} \quad (58c)$$

As we have done before, we can perform a line integral to define the swept area at the time the right-hand well breaks through. It is,

$$(E_a)_{-right} = \frac{2\pi C}{C} \int_0^C \frac{(-x^3 - Cx^2 + 2C^2x) dx}{(2C-x)} \quad (59a)$$

After considerable algebra, Eq. 59a simplifies to,

$$(E_a)_{-right} = \frac{2\pi}{C} \left[\int_0^C x^2 dx + 3C \int_0^C x dx + 4C^2 \int_0^C dx + 8C^3 \int_0^C \frac{dx}{x-2C} \right] \quad (59b)$$

Equation 78b, when evaluated at the limits, becomes,

$$\begin{aligned} (E_a)_{-right} &= \frac{2\pi}{C} \left[\frac{C^3}{3} + \frac{3C^3}{2} + 4C^3 - 8C^3 \ln 2 \right] \\ &= \frac{2\pi}{C} \left(\frac{35}{6} - 8 \ln 2 \right) C^3 = 2\pi C^2 (0.288156) \\ &= 0.576312\pi C^2 \end{aligned} \quad (59c)$$

It is interesting to compare the area swept for this system to that for a balanced three-spot and a balanced four-spot. We would expect the area swept to be greater than it is for a balanced three-spot since the other well is further away and since only 1/3 of the injected fluid moves toward the right-hand producer.

A balanced four-spot also has only 1/3 of the total fluid moving toward the right-hand producer, but its left-hand geometry differs from this figure. In this figure, the right-hand producer is closer to the injection well than the left-hand producer. Because of this difference in geometry, we would expect this breakthrough to be a bit sooner than it is for a balanced four-spot pattern.

These expectations are borne out. A balanced three-spot pattern has a breakthrough sweep of $0.50000\pi C^2$ which is smaller than Eq. 59c as expected. A balanced four-spot has a breakthrough sweep of $0.60000\pi C^2$ which is slightly larger than Eq. 59c as we thought it might be. It is surprising that the balanced four-spot area sweep is so close to that of this pattern.

A further idea may occur to the more thoughtful reader. Since 1/3 of the total injection moves to the right, and 2/3 to the left, it might seem that a straight line at 60° would act as dividing streamline for this system. This idea is quite logical--but incorrect. It is correct very close to the injection well but veers away from a straight line as x or y become larger. I have not shown the mathematical proof of this statement here. The interested reader can work out the math for himself to see if I am correct.

Next let us consider the movement to the left in an unbalanced inverted three-spot pattern. We will now define x to be the distance to the left of the injector. The pressure equation is,

$$\frac{4\pi kh}{q\mu} [p(x,0) - p_i] = -\ln(x)^2 + \frac{1}{3}\ln(C+x)^2 + \frac{2}{3}\ln(2C-x)^2 \quad (60)$$

and when differentiated with respect to x , Eq. 60 becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = -\frac{1}{x} + \frac{1}{3(C+x)} - \frac{2}{3(2C-x)} \quad (61a)$$

$$= \frac{-3(C+x)(2C-x) + x(2C-x) - 2x(C+x)}{3x(C+x)(2C-x)} \quad (61b)$$

which after some algebra, becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p}{\partial x} = \frac{-C(2C+x)}{x(C+x)(2C-x)} = \frac{-C(2C+x)}{(-x^3 + Cx^2 + 2C^2x)} \quad (61c)$$

As before, we can now define the breakthrough sweep to the left as a line integral. It is,

$$(E_a)_{-left} = \frac{2\pi}{C} \left[-\int_0^{2C} \frac{(-x^3 + Cx^2 + 2C^2x) dx}{(2C+x)} \right] \quad (62a)$$

After considerable algebraic manipulation, Eq. 62a becomes,

$$(E_a)_{-left} = \frac{2\pi}{C} \left[-\int_0^{2C} x^2 dx + 3C \int_0^{2C} x dx - 4C^2 \int_0^{2C} dx + 8C^3 \int_0^{2C} \frac{dx}{(2C+x)} \right] \quad (62b)$$

which, when integrated becomes,

$$(E_a)_{-left} = \frac{2\pi}{C} \left[-\frac{x^3}{3} + \frac{3Cx^2}{2} - 4C^2x + 8C^3 \ln(2C+x) \right]_0^{2C} \quad (62c)$$

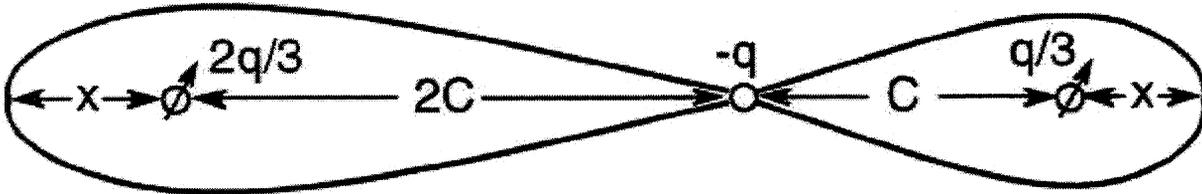
$$(E_a)_{-left} = \frac{2\pi}{C} \left(-\frac{8C^3}{3} + 6C^3 - 8C^3 + 8C^3 \ln 2 \right)$$

$$= \frac{2\pi}{C} \left(-\frac{14}{3} + 8 \ln 2 \right) C^3 = 1.757022\pi C^2 \quad (62d)$$

Notice that the left-hand producer breaks through far later than the right-hand producer . It doesn't take four times as long, as we might have first guessed since the distance is twice as far. It actually takes about three times as long to break through.

4.4 UNEQUAL RATES (NORMAL THREE-WELL SYSTEM)

We can also look at exactly the same geometry as previously, but with the well rates inverted. That is, we can have injection rates of $2q/3$ and $q/3$ at the two outer wells, and production rate at $-q$ at the center well, as shown in this attached sketch.



There is no need to calculate the areas swept at breakthrough for this system. They will be the same as they were for the inverted pattern we just discussed. We can, however, calculate the distances moved to the right of the right-hand injector and to the left of the left-hand injector along the horizontal tear drops.

For the right-hand injector, the pressures to the right beyond the well can be written as follows,

$$\frac{4kh[p(x,0) - p_i]}{q\mu} = -\frac{1}{3} \ln(x)^2 - \frac{2}{3} \ln(3C+x)^2 + \ln(C+x)^2 \quad (63)$$

When Eq. 63 is differentiated with respect to x to define the velocity function, it becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{1}{3x} - \frac{2}{3(3C+x)} + \frac{1}{C+x} \quad (64a)$$

which simplifies to,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = \frac{-3C^2 + 3Cx}{3(3C^2x + 4Cx + x^3)} = \frac{-C(C-x)}{x(C+x)(3C+x)} \quad (64b)$$

Equation 64b can be integrated and its time set equal to the time to breakthrough of the right-hand well, Eq. 59c.

The result is,

$$2\pi C^2(0.288156) = \frac{2\pi}{C} \int_0^x \frac{(x^3 + 4Cx^2 + 3C^2x) dx}{(C-x)} \quad (65a)$$

which after considerable algebra, simplifies to

$$0.288156C^2 = \frac{1}{C} \left[-\int_0^x x^2 dx - 5C \int_0^x x dx - 8C^2 \int_0^x dx + 8C^3 \int_0^x \frac{dx}{(C-x)} \right] \quad (65b)$$

which, when integrated, becomes,

$$0.288156C^2 = \frac{1}{C} \left[-\frac{x^3}{3} - \frac{5Cx^2}{2} - 8C^2x - 8C^3 \ln\left(\frac{C-x}{C}\right) \right] \quad (65c)$$

which, after further algebra, becomes,

$$0.288156 = -\frac{1}{6} \left(\frac{x}{C} \right) \left[2 \left(\frac{x}{C} \right)^2 + 15 \left(\frac{x}{C} \right) + 48 \right] + 8 \ln \left(\frac{1}{1 - \frac{x}{C}} \right) \quad (65d)$$

Equation 65d must be solved by trial and error. Its solution is,

$$\frac{x}{C} = 0.33388 \quad (66)$$

It is interesting to compare this distance moved to breakthrough to the distances moved in the tear drop shapes of the isolated normal three-spot and four-spot patterns. For the three-spot, the distance moved is 0.41421, from Eq. 22b, p. 72. For the four-spot, the distance moved is 0.35721, from Eq. 36, p. 76. Note that both these results are larger than the value of 0.33388 from Eq. 66. This result is interesting, for in the isolated four-spot, the area swept is also slightly greater than it is for the unbalanced pattern we've been discussing here. Thus their shapes will be quite similar. These

differences are rather modes--they differ by considerably less than 10 percent. So, as a first approximation, one can assume the simpler inverted four-spot equation is approximately correct.

Finally, we would like to look at the distance moved to the left of the left-hand injector at the time it breaks through. This concept is illustrated by the x on the left in the sketch of the system. The pressure equation for this x location is as follows,

$$\frac{4\pi kh[p(x,0) - p_i]}{q\mu} = -\frac{2}{3}\ln(x)^2 - \frac{1}{3}\ln(3C+x)^2 + \ln(2C+x)^2 \quad (67)$$

When Eq. 67 is differentiated with respect to x , it becomes,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{2}{3x} - \frac{1}{3(3C+x)} + \frac{1}{2C+x} \quad (68a)$$

which simplifies to,

$$\frac{2\pi kh}{q\mu} \frac{\partial p(x)}{\partial x} = -\frac{-4C^2 - Cx}{6C^2x + 5Cx + x^3} = \frac{-C(4C+x)}{x(2C+x)(3C+x)} \quad (68b)$$

As we have done before, we can write a line integral for the movement along this streamline, and set the time equal to the breakthrough time to this well, Eq. 62d. The resulting equation is,

$$1.757022\pi C^2 = \frac{2\pi}{C} \int_0^x \frac{x^3 + 5Cx^2 + 6C^2x}{(x+4C)} dx \quad (69a)$$

which, after some algebra, becomes,

$$0.878511C^2 = \frac{1}{C} \left[\int_0^x x^2 dx + C \int_0^x x dx + 2C^2 \int_0^x dx - 8C^3 \int_0^x \frac{dx}{x+4C} \right] \quad (69b)$$

which, when integrated becomes,

$$0.878511C^2 = \frac{1}{C} \left[\frac{x^3}{3} + \frac{Cx^2}{2} + 2C^2x - 8C^3 \ln\left(\frac{x+4C}{4C}\right) \right] \quad (69c)$$

and, after further algebra, the equation becomes,

$$0.878511 = \frac{1}{6} \left(\frac{x}{C} \right) \left[2 \left(\frac{x}{C} \right)^2 + 3 \left(\frac{x}{C} \right) + 12 \right] - 8 \ln \left(\frac{4 + \frac{x}{C}}{4} \right) \quad (69d)$$

Equation 69d must be solved by trial and error. Its solution is,

$$\frac{x}{C} = 0.92552 \quad (70)$$

We should find it interesting to compare this distance to the distances moved for a two-spot (double) and a balanced three-spot pattern. From the geometry and rates in the system, we would expect that behavior to lie between these two patterns. Further, we should expect the result of Eq. 70 to lie nearer to the two-spot than the three-spot, since the right-hand well would be expected to have a minor effect. This supposition is found to be correct. In Eq. 5e, p. 66, of these notes the value x/C was 0.5000. But remember that the spacing between the wells was only C rather than $2C$ as it is here. So the equivalent distance is 1.0000, slightly greater than Eq. 70, as we speculated.

For the balanced three-spot the distance, from Eq. 22b, p. 72, is $x/C = 0.41421$. Again we should double this number for the doubled spacing, with the result that the constant is 0.82843, smaller than Eq. 70 as we speculated. Further, as we speculated, the behavior of the left hand side of this pattern is somewhat closer to the doublet behavior than it is to the three-spot.

4.5 CONCLUDING REMARKS ON ISOLATED PATTERNS

We have looked at a host of balanced patterns at unity mobility ratio in this set of notes. The geometries and rates ranged broadly. But we found that, whenever total production and injection are equal, we can gain considerable insight on the flow equations and the fluid movement.

The balanced patterns, where the wells are arrayed around a single injector or a single producer, we found that simple equations can define the nature of the steady state flow lines and geometries of the flow paths, and their breakthrough behavior. When the rates are not equal, but still are balanced, the geometries are more complex, but still amenable to analytic solution.

It is common to treat problems of this type using finite difference computer calculations. Often this is not necessary; and also, often such calculations have been found to be in error because of the sizes of grid blocks necessary. The insight these analytic expressions give us can be useful, in themselves, to solve real problems; and they also can be used as checks to assess the accuracy of finite difference calculations.

5. DEPLETING RADIAL SYSTEMS WITH OFF-CENTER WELLS

A depleting radial system, where the well is off center, is not a typical doublet problem; for the image function is producing rather than injecting. However, the production rate from the image function is the same as the rate inside the circle, so the resulting mathematics has many similarities to the doublet problems we have been discussing at some length here.

This problem was addressed by Kwaku Temeng in 1982 as part of his Engineer's Degree thesis, and later published with his advisor, Roland Horne (SPEJ, Dec. 84, pp. 677-684). A clear definition of the geometry and important variables for his problem is shown in his Fig. 1 on the next page. There was a slight error in his figure. The equation on the upper right should have shown D_D rather than D . I've written this here. He labels all the distance variables as dimensionless, based on r_w . So the variables themselves have the same relation to each other as the dimensionless variables seen in this figure.

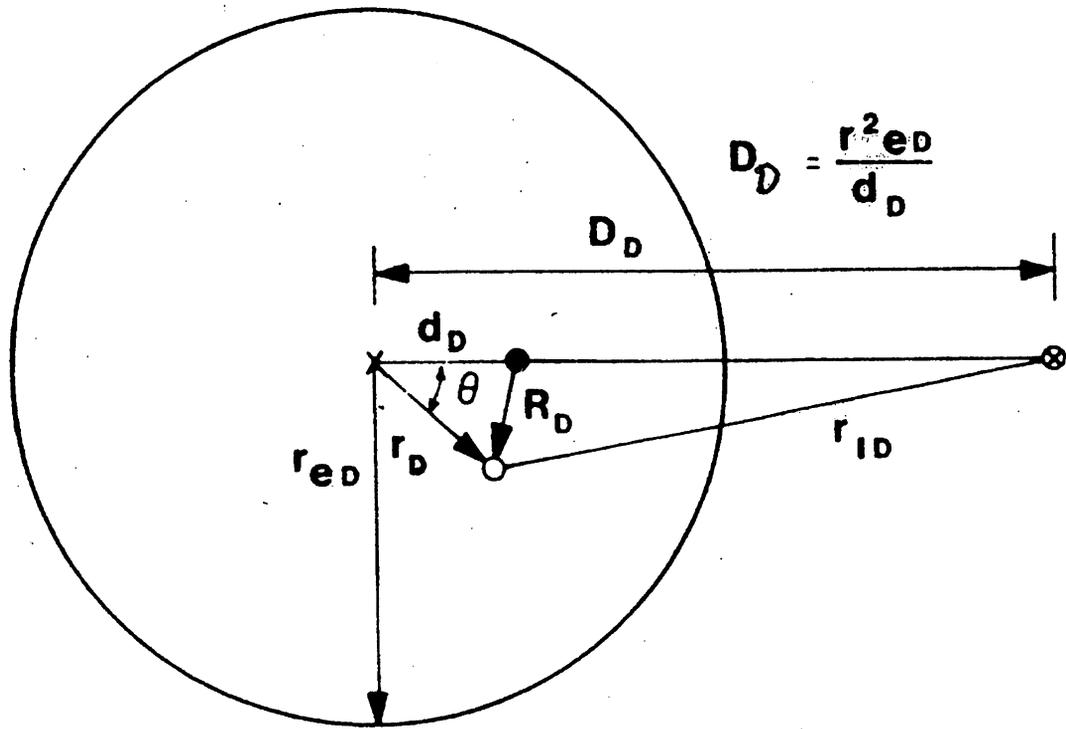
When a system is being depleted, the outer boundary must be a no-flow boundary. They show the solution for this case in their Eq. 28, partially repeated below.

$$p_D(r, t_D) = 2\pi t_{DA} + \frac{r_D^2 + d_D^2}{2(r_e)_D^2} - \frac{3}{4} - \ln \left[\frac{R_D d_D (r_1)_D}{(r_e)_D^3} \right] - \Sigma \Sigma \quad (1a)$$

Note the double summation sign at the end of Eq. 1a. The complete listing of this summation is a very complex expression that becomes zero once pseudosteady state is reached. That is, whenever each part of the system starts to deplete at a constant rate when the well is being produced at a constant rate. Well only be looking at that condition in these notes, so henceforth, I'll eliminate the summation terms. Note that Eq. 1a can be written using the actual distance variables, so I'll do that to make the later manipulations easier.

$$[p_D(r, t_D)]_{pss} = 2\pi t_{DA} + \frac{r^2 + d^2}{2r_e^2} - \frac{3}{4} - \ln \left(\frac{R d r_1}{r_e^3} \right) \quad (1b)$$

It might be worthwhile to briefly discuss the variables in Eq. 1b. Of course, p_D and t_{DA} are well known and obvious. The variable, r , is at any point within the radial system, defined by the distance from the center, r , and



● WELL POINT

○ OBSERVATION POINT

x CENTER OF CIRCLE

⊗ INVERSE POINT OF WELL
IN CIRCLE

Fig. 1—Geometric representation of circular system.

angle, θ . The term, d , is the distance from the center of the depleting system to the producing well location, and R is the distance from the well to the general point defined by r . The depletion boundary is at r_e . The radius, r_1 , is the distance from the general point, r , to the image well shown in Fig. 1. The distance to the image well, D , is shown to be,

$$D = r_e^2 / d \quad (2)$$

from the center of the circle through the well at d . This is exactly the same point as the image location for a well which causes a constant pressure circle, shown in Section 3.3 of these notes by Eq. 6a, p. 42. The equations are expressed somewhat differently, but they define the same point.

Equation 1b defines the pressure at any point in the circular region. If we wish to evaluate this equation at the producing well, then r becomes d , r_1 becomes $D - d$, and R becomes r_w . Thus Eq. 1b simplifies to,

$$(p_w)_D = 2\pi t_{DA} + \frac{d^2}{r_e^2} - \frac{3}{4} - \ln \left[\frac{r_w d (D - d)}{(r_e)^3} \right] \quad (3a)$$

Substituting Eq. 2 into Eq. 3a, we get,

$$(p_w)_D = 2\pi t_{DA} + \frac{d^2}{r_e^2} - \frac{3}{4} - \ln \left[1 - \left(\frac{d}{r_e} \right)^2 \right] + \ln \left(\frac{r_e}{r_w} \right) \quad (3b)$$

Let's consider Eq. 3b in more detail, by comparing it to the equation we would get if the well were centered in the circle. First we should realize that,

$$(\bar{p}_A)_D = 2\pi t_{DA} \quad (4)$$

This is the general depletion material balance. We can now insert Eq. 4 into Eq. 3b and rearrange to get,

$$\begin{aligned} (p_w)_D - (\bar{p}_A)_D &= -\frac{2\pi kh(\bar{p} - p_w)}{q_w \mu} \\ &= \frac{d^2}{r_e^2} - \frac{3}{4} - \ln \left[1 - \left(\frac{d}{r_e} \right)^2 \right] + \ln \left(\frac{r_e}{r_w} \right) \end{aligned} \quad (5)$$

My notes on pseudosteady state flow (Brigham, 1988) show that for a radial system with the well in the center, the equation is,

$$-\frac{2\pi kh(\bar{p} - p_w)}{q_w \mu} = \frac{r_e^2 \ln(r_e/r_w)}{(r_e^2 - r_w^2)} - \frac{3r_e^2 - r_w^2}{4(r_e^2 - r_w^2)} \quad (6a)$$

If we assume that the well radius is negligible compared to the outer boundary radius, as Temeng did, then Eq. 6a reduces to,

$$-\frac{2\pi kh(\bar{p} - p_w)}{q_w \mu} = \ln\left(\frac{r_e}{r_w}\right) - \frac{3}{4} \quad (6b)$$

Note that the term, $\ln(r_e/r_w) - 3/4$, is included in Eq. 5. Thus the additional terms, $d^2/r_e^2 - \ln[1 - (d/r_e)^2]$, are an indication of the additional pressure drop due to the well being off center in the system. Also note that as the off center distance, d , approaches zero, Eq. 5 becomes Eq. 6b.

It might be informative to compare this result with that of an off center well with a constant pressure outer boundary. Section 3.3 of these notes, Eq. 18, p. 43, shows that when the well is off center, the pressure drop is decreased due to the effect of the term, $\ln[1 - (d/r_e)^2]$. So in the constant pressure system a well off center causes a smaller pressure drop, while in a depleting system it causes a greater pressure drop. All of these results make sense logically. Also, notice that similar d/r_e terms are involved, which too, seems logical.

Let's return now to discussion of Eqs. 1a and 1b. Nowhere in Temeng's thesis or paper does he show how this equation was derived. And I'm certainly not going to try to derive it here. However, we should be able to prove that it fits the no-flow boundary condition when $r = r_e$. To do this, we first need to define some of the variables in terms of r , as follows.

$$R^2 = r^2 + d^2 - 2rd \cos \theta \quad (7)$$

and

$$(r_1)^2 = r^2 + D^2 - 2rD \cos \theta \quad (8)$$

Also, for convenience, we should square these terms in Eq. 1b. The result is thus,

$$[p_D(r, t_D)]_{pss} = 2\pi t_{DA} + \frac{r^2 + d^2}{2r_e^2} - \frac{3}{4} - \ln\left[\frac{d}{(r_e)^3}\right] - \frac{1}{2} \ln(Rr_1)^2 \quad (1c)$$

To prove the exterior is a no-flow boundary, we need to differentiate Eq. 1c and set it equal to zero at r_e . Upon differentiating, we get,

$$\left[\frac{\partial p_D(r, t_D)}{\partial r}\right]_{pss} = \frac{\partial(r^2 + d^2)}{2r_e^2 \partial r} - \frac{1}{2} \frac{\partial}{\partial r} [\ln(Rr_1)^2] \quad (9)$$

Of course the first term on the right of Eq. 9 is easy to evaluate at r_e . It is

$$\left[\frac{\partial}{\partial r} \left(\frac{r^2 + d^2}{2r_e^2}\right)\right]_{r=r_e} = \left(\frac{r}{r_e^2}\right)_{r=r_e} = \frac{1}{r_e} \quad (10)$$

The second term requires more effort.

In general we can write the expression for the derivative of the second term as follows,

$$\left[\frac{\partial \ln(R^2 r_1^2)}{\partial r}\right]_{\theta} = \left[\frac{r_1^2 (\partial R^2) + R^2 (\partial r_1^2)}{R^2 r_1^2 \partial r}\right]_{\theta} \quad (11)$$

Note that I will be evaluating these terms at a constant angle, θ . Differentiating R^2 , from Eq. 7, we get,

$$\left(\frac{\partial R^2}{\partial r}\right)_{\theta} = 2r - 2d \cos \theta \quad (12)$$

and from Eq. 8, we get,

$$\left[\frac{\partial (r_1)^2}{\partial r}\right]_{\theta} = 2r - 2D \cos \theta \quad (13)$$

Substituting Eqs. 7, 8, 12 and 13 into Eq. 11, we get,

$$\left[\frac{\partial \ln(R^2 r_1^2)}{\partial r}\right]_{\theta, r} = \frac{(r^2 + D^2 - 2rD \cos \theta)(2r - 2d \cos \theta) + (r^2 + d^2 - 2rd \cos \theta)(2r - 2D \cos \theta)}{(r^2 + d^2 - 2rd \cos \theta)(r^2 + D^2 - 2rD \cos \theta)} \quad (14)$$

We wish to evaluate Eq. 14 when $r = r_e$, so we get,

$$\left[\frac{\partial \ln(R^2 r_1^2)}{\partial r} \right]_{\theta, r_e} = \frac{(r_e^2 + D^2 - 2r_e D \cos \theta)(2r_e - 2d \cos \theta) + (r_e^2 + d^2 - 2r_e d \cos \theta)(2r_e - 2D \cos \theta)}{(r_e^2 + d^2 - 2r_e d \cos \theta)(r_e^2 + D^2 - 2r_e D \cos \theta)} \quad (15a)$$

$$\begin{aligned} & 2r_e^3 - 2dr_e^2 \cos \theta + 2r_e D^2 - 2D^2 d \cos \theta - 4r_e^2 D \cos \theta \\ & + 4r_e D d \cos^2 \theta + 2r_e^3 - 2r_e^2 D \cos \theta + 2r_e d^2 - 2D d^2 \cos \theta \\ \left[\frac{\partial \ln(R^2 r_1^2)}{\partial r} \right]_{\theta, r_e} &= \frac{-4r_e^2 d \cos \theta + 4r_e d D \cos^2 \theta}{r_e^4 + r_e^2 D^2 - 2r_e^3 D \cos \theta + r_e^2 d^2 + d^2 D^2 - 2r_e D d^2 \cos \theta} \\ & - 2r_e^3 d \cos \theta - 2r_e d D^2 \cos \theta + 4r_e^2 d D \cos^2 \theta \end{aligned} \quad (15b)$$

Several times in Eq. 15b there is a dD multiplier, which from Eq. 2 is equal to r_e^2 . We'll substitute this concept into

Eq. 15b to get,

$$\begin{aligned} & 2r_e^3 - 2dr_e^2 \cos \theta + 2r_e D^2 - 2Dr_e^2 \cos \theta - 4r_e^2 D \cos \theta \\ & + 4r_e^3 \cos^2 \theta + 2r_e^3 - 2r_e^2 D \cos \theta + 2r_e d^2 - 2r_e^2 d \cos \theta \\ \left[\frac{\partial \ln(R^2 r_1^2)}{\partial r} \right]_{\theta, r_e} &= \frac{-4r_e^2 d \cos \theta + 4r_e^3 \cos^2 \theta}{r_e^4 + r_e^2 D^2 - 2r_e^3 D \cos \theta + r_e^2 d^2 + r_e^4 - 2r_e^3 d \cos \theta} \\ & - 2r_e^3 d \cos \theta - 2r_e^3 D \cos \theta + 4r_e^4 \cos^2 \theta \end{aligned} \quad (15c)$$

which simplifies greatly to,

$$\begin{aligned} &= \frac{4r_e^3 - 8r_e^2 d \cos \theta + 2r_e D^2 + 2r_e d^2 - 8r_e^2 D \cos \theta + 8r_e^3 \cos^2 \theta}{2r_e^4 + r_e^2 D^2 + r_e^2 d^2 - 4r_e^3 D \cos \theta - 4r_e^3 d \cos \theta + 4r_e^4 \cos^2 \theta} \\ &= \frac{2r_e \left[2r_e^2 + D^2 + d^2 - 4r_e(D+d)\cos \theta + 4r_e^2 \cos^2 \theta \right]}{r_e^2 \left[2r_e^2 + D^2 + d^2 - 4r_e(D+d)\cos \theta + 4r_e^2 \cos^2 \theta \right]} \\ \left[\frac{\partial \ln(R^2 r_1^2)}{\partial r} \right]_{\theta, r_e} &= 2/r_e \end{aligned} \quad (15d)$$

Substituting Eqs. 10 and 15d into Eq. 9, we get,

$$\left[\frac{\partial p_D(r, t_D)}{\partial r} \right]_{\theta, r_e}^{pss} = \frac{1}{r_e} - \frac{1}{2} \left(\frac{2}{r_e} \right) = 0 \quad (16)$$

So the gradient is zero at $r = r_e$ for any angle, θ , just as it should be. This confirms that Eq. 1b fits the boundary conditions for a depleting system, and is the pseudosteady state solution for this system.

It should be of interest to determine the pressure profiles along the boundary as a function of the boundary angle and the location of the producing well. Temeng did not do this, although his equations show us how to evaluate these pressures. It should be clear that, as the producing well moves off center, the boundary pressures near that well will be lower than in other parts of the perimeter, and thus can be lower than the average pressure. Meanwhile, the boundary pressures in the opposite direction away from the near boundary will be higher than the average pressure. It will be instructive to calculate these pressures for various well locations. To accomplish this, we will first need to perform some algebra.

Combining Eqs. 4 and 1c, we get,

$$p_D(r, t_D) - (\bar{p}_D)_A = \frac{r^2 + d^2}{2r_e^2} - \frac{3}{4} - \frac{1}{2} \ln \left[\frac{R^2 d^2 (r_1)^2}{(r_e)^6} \right] \quad (17)$$

In Eqs. 7 and 8 I defined R^2 and $(r_1)^2$, which I'll repeat below, setting r equal to r_e ,

$$R^2 = r_e^2 + d^2 - 2r_e d \cos \theta \quad (18)$$

$$(r_1)^2 = r_e^2 + D^2 - 2r_e D \cos \theta \quad (19)$$

We'll use Eq. 2 to eliminate the variable, D , from Eq. 19,

$$(r_1)^2 = r_e^2 + r_e^4 / d^2 - 2(r_e^3 / d) \cos \theta \quad (20)$$

Now, let us substitute Eqs. 18 and 20 into the argument of the log term in Eq. 17,

$$\frac{R^2 d^2 (r_1)^2}{(r_e)^6} = \frac{d^2}{r_e^6} (r_e^2 + d^2 - 2r_e d \cos \theta) \left[r_e^2 + \frac{r_e^4}{d^2} - 2 \left(\frac{r_e^3}{d} \right) \cos \theta \right] \quad (21a)$$

$$= \frac{d^2}{r_e^6} \left[r_e^4 + \frac{r_e^6}{d^2} - \frac{2r_e^5}{d} \cos \theta + r_e^2 d^2 + r_e^4 - 2r_e^3 d \cos \theta - 2r_e^3 d \cos \theta - \frac{2r_e^5}{d} \cos \theta + 4r_e^4 \cos^2 \theta \right] \quad (21b)$$

$$= \left(\frac{d}{r_e} \right)^2 + 1 - 2 \left(\frac{d}{r_e} \right) \cos \theta + \left(\frac{d}{r_e} \right)^4 + \left(\frac{d}{r_e} \right)^2 - 2 \left(\frac{d}{r_e} \right)^3 \cos \theta - 2 \left(\frac{d}{r_e} \right)^3 \cos \theta - 2 \left(\frac{d}{r_e} \right) \cos \theta + 4 \left(\frac{d}{r_e} \right)^2 \cos^2 \theta \quad (21c)$$

$$\begin{aligned}
&= 1 + 2\left(\frac{d}{r_e}\right)^2 + \left(\frac{d}{r_e}\right)^4 - 4\left(\frac{d}{r_e}\right)\cos\theta - 4\left(\frac{d}{r_e}\right)^3\cos\theta + 4\left(\frac{d}{r_e}\right)^2\cos^2\theta \\
&= \left[1 + \left(\frac{d}{r_e}\right)^2\right]^2 - 4\left[1 + \left(\frac{d}{r_e}\right)^2\right]\left[\left(\frac{d}{r_e}\right)\cos\theta + \left(\frac{d}{r_e}\right)^2\cos^2\theta\right] \\
&= \left[1 + \left(\frac{d}{r_e}\right)^2 - 2\left(\frac{d}{r_e}\right)\cos\theta\right]^2
\end{aligned} \tag{21d}$$

Thus the log term in Eq. 17 simplifies to,

$$-\frac{1}{2}\ln\left[\frac{R^2 d^2 (r_1)^2}{(r_e)^6}\right] = -\ln\left[1 - 2\left(\frac{d}{r_e}\right)\cos\theta + \left(\frac{d}{r_e}\right)^2\right] \tag{22}$$

This is a remarkable simplification which will make subsequent calculations much easier to perform.

We can also simplify the remaining terms on the right-hand side of Eq. 17. When $r = r_e$ we get,

$$\left[\frac{r^2 + d^2}{2r_e^2} - \frac{3}{4}\right]_{r=r_e} = \frac{r_e^2 + d^2}{2r_e^2} - \frac{3}{4} = \frac{1}{2}\left(\frac{d}{r_e}\right)^2 - \frac{1}{4} \tag{23}$$

So the total result changes Eq. 17 to the following, when $r = r_e$.

$$[p_D(r_e, t_D) - (\bar{p}_A)_D]_{d, \theta} = \frac{1}{2}\left(\frac{d}{r_e}\right)^2 - \frac{1}{4} - \ln\left[1 - 2\left(\frac{d}{r_e}\right)\cos\theta + \left(\frac{d}{r_e}\right)^2\right] \tag{24}$$

It would be useful to compare Eq. 24 for a well off center, to the result for a centered well, where $d = 0$. Under this condition, from Eq. 24, we get,

$$[p_D(r_e, t_D) - (\bar{p}_A)_D]_{d=0} = -\frac{1}{4} \tag{25}$$

Thus on subtracting Eq. 25 from Eq. 24, to compare the results, we get the following simple equation,

$$[p_D(r_e)_{d, \theta} - p_D(r_e)_{d=0}]_{pss} = \frac{1}{2}\left(\frac{d}{r_e}\right)^2 - \ln\left[1 - 2\left(\frac{d}{r_e}\right)\cos\theta + \left(\frac{d}{r_e}\right)^2\right] \tag{26}$$

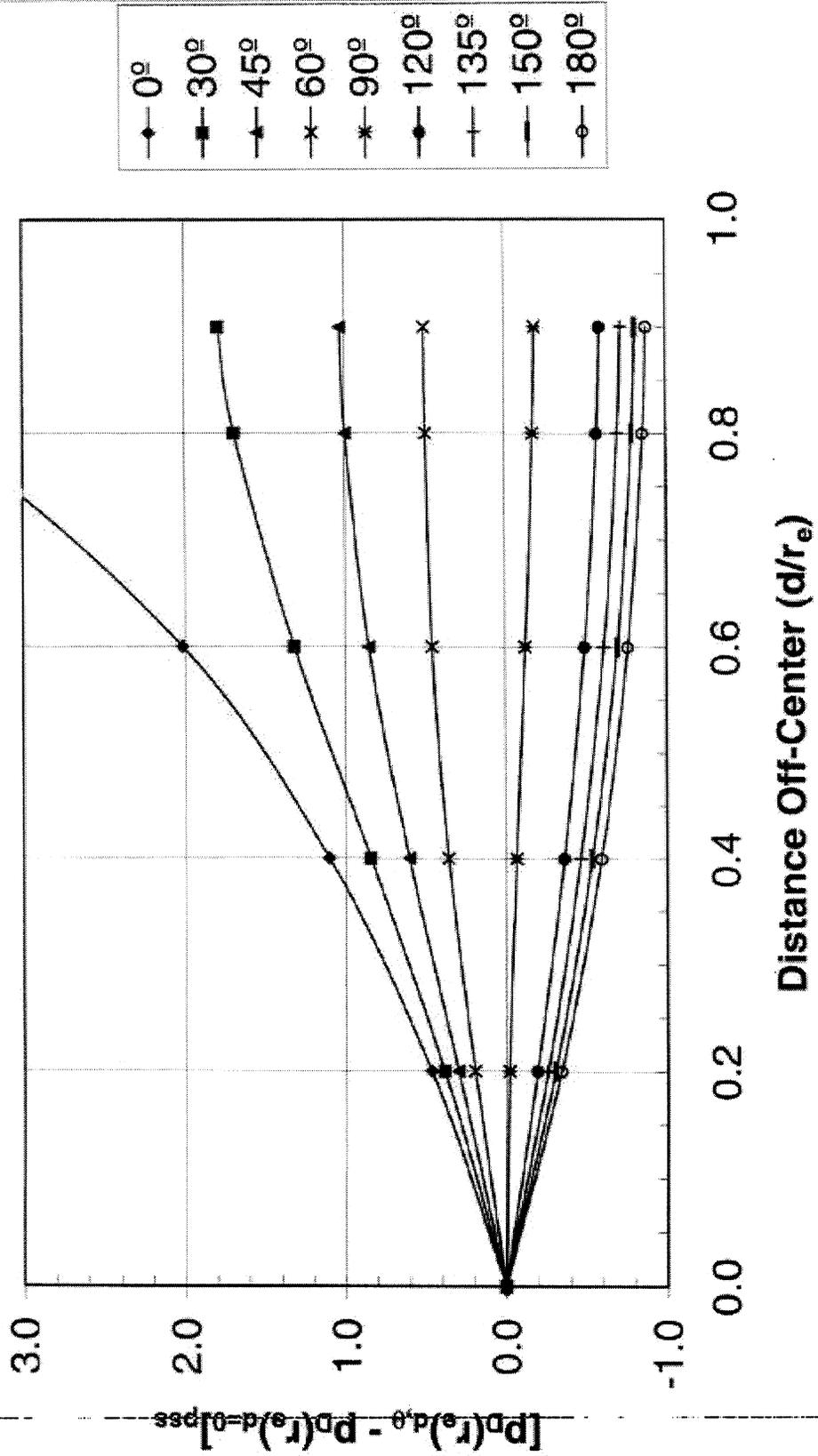
We will evaluate Eq. 26 at various angles, and at various off-center positions, in the following two tables, and also graph the results on p. 107 to help clarify the picture of the resulting “tilted” pressure boundary. I use simple regular angles, for these calculations: $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ, 135^\circ, 150^\circ$, and 180° .

**Dimensionless Pressure Drops at Various Angles
For Wells Off-Center in Depleting Systems**

Distance off Center, d/r_e	$[p_D(r_e)_{d,\theta} - p_D(r_e)_{d=0}]_{pss}$ Equation 26				
	$\theta = 0^\circ$	$\theta = 30^\circ$	$\theta = 45^\circ$	$\theta = 60^\circ$	$\theta = 90^\circ$
0	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.46629	0.38587	0.29818	0.19435	-0.01922
0.4	1.10165	0.84104	0.60035	0.35449	-0.06842
0.6	2.01258	1.31703	0.85047	0.45444	-0.12748
0.8	3.53888	1.68900	0.99603	0.49435	-0.17470
0.9	5.01019	1.78670	1.02637	0.49931	-0.18833

	$\theta = 120^\circ$	$\theta = 135^\circ$	$\theta = 150^\circ$	$\theta = 180^\circ$
0	0.00000	0.00000	0.00000	0.00000
0.2	-0.19511	-0.25978	-0.30672	-0.34464
0.4	-0.36464	-0.46563	-0.53671	-0.59294
0.6	-0.49294	-0.61233	-0.69515	-0.76001
0.8	-0.57200	-0.69934	-0.78711	-0.85557
0.9	-0.59195	-0.72068	-0.80957	-0.87871

Boundary Pressures for an Off-Center Well in a Depleting Radial System



The attached graph is worth discussing further. First I'll discuss the signs on the numerical results. From the definition of the dimensionless pressures, they are negative when the well is a producer. As a result, the negative signs in the table and graph indicate pressure higher than the pressure one would get if the well were centered, and positive signs indicate pressures lower than well-centered boundary pressures. Clearly, from these results, it is obvious that the pressures at boundary locations near the well are considerably below those we would get if the well were centered, and are even below the average pressure, as we should expect, based on the geometry of the system. Similarly, pressures at boundary locations farthest from the well are higher than for the centered well boundary pressures. Also it is clear that these effects become much stronger as the producing well approaches the circular boundary. All these results are quite logical, upon reflection.

In the doublet system, for a producing well off-center in a circle, we found that the constant pressure loci were also circles. This concept was discussed in Section 1 of these notes on pp. 1-5. One might think that this same concept would hold true for the equal pressure loci of the depleting system. It does not! These equal pressure shapes are quite complex, and I have not tried to evaluate them for that reason.

6. CONCLUSIONS

The derivations here show that, when there is a pair of wells injecting and producing at the same rate, after a short period of time the equal pressure loci are circles. Also the flow lines are circles. Muscat showed these results many decades ago. The knowledge of these geometries allows us to calculate the breakthrough times for such doublet systems, the post-breakthrough behavior and the detailed areal sweep behavior of doublets.

Using the concepts from this doublet behavior, a number of other problems useful to petroleum engineers can be solved. I show the solutions for systems that gradually become more complex: starting with a well near a constant pressure boundary, then the heat loss equation for buried pipelines, the behavior of a well off center in a constant pressure circle, heat loss for a steam injection well with insulation in the tubing/casing annulus, and finally, the pressure behavior for multiple wells in a constant pressure circle. All these results followed directly from the doublet concepts.

A number of multi-well problems are addressed where the total injection and production are the same. Various simple patterns exhibit logical behavior as more wells are added around a central well. The resulting recurrence relations are quite simple and logical, even though the intermediate equations leading up to them can become quite complex. Other more complex balanced systems are also looked at briefly to illuminate the mathematics involved. Although these latter systems are more complex; they are still amenable to analytical analysis.

Finally, I've also discussed the behavior of a depleting radial system with an off-center well. This is not a true doublet problem, but it has many characteristics that are similar to doublet mathematics. The exterior boundary pressure profiles on such systems are shown as a function of the off-center distance. These pressures are tilted, with the lowest pressures at the boundary near the well, and the highest pressures away from the well.

All the results shown here are exact analytic solutions. Often problems of this nature are addressed using finite difference calculations, and often this isn't necessary. Further the finite difference results are often in error due to discretization problems near the wells. Thus these analytic problems are also useful to assess the accuracy of finite difference calculations.

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